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Strongly positive representations of metaplectic groups

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ABSTRACT

In this paper, we obtain a classification of irreducible strongly positive square-integrable genuine representations of metaplectic groups over p-adic fields, using a purely algebraic approach. Our results parallel those of Mœglin and Tadić for classical groups, but their work relies on certain conjectures. On the other hand, our results are complete and there are no additional conditions or hypotheses. The important point to note here is that our results and techniques can be used in the case of classical p-adic groups in a completely analogous manner.

1. Introduction

Admissible representations of metaplectic groups over p-adic fields have recently been intensively studied by many authors and many results, mostly similar to those related to the representation theory of classical groups [3,4,7], have been achieved. It is of particular interest to obtain knowledge about the square-integrable representations of metaplectic groups, especially about the irreducible ones, the so-called discrete series. In the papers [11,12], Mœglin and Tadić have classified discrete series of classical groups over p-adic fields, assuming certain conjectures. It is of interest to know whether there is an analogous classification for metaplectic groups and whether their assumptions may be removed. The aim of this paper is to address these problems for an important type of square-integrable representation, namely the strongly positive ones, which can be viewed as basic building blocks for all the square-integrable representations. Important examples of strongly positive square-integrable representations are generalized Steinberg representations and regular discrete series, which have been classified by Tadić in [19]. In the Mœglin–Tadić classification, strongly positive discrete series correspond to so-called alternating triples.

The main difficulty in carrying out their construction for the case of metaplectic groups is that the work of Mœglin [11] relies on the theory of L-functions, which we do not have at our disposal in its

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full generality. Instead of extending this theory to the metaplectic case, or using the very powerful methods of theta-correspondence, we classify strongly positive discrete series in completely algebraic way. The starting point of our approach is the analysis of certain useful embeddings of irreducible representations, which were introduced first in [15] and further developed in [6]. We use mostly basic techniques and our classification involves no hypotheses. This approach provides a rather combinatorial algorithm for constructing the classifying data, which should be useful in other contexts, such as calculations with Jacquet modules. The results of this paper may be straightforwardly extended to the case of classical groups. Further, such a classification allows one to study composition series of some generalized principal series of metaplectic groups, as has been done in [14] in the case of classical groups.

Now we describe the contents of the paper, section by section.

In the next section we set up notation and terminology, while the third section is devoted to the study of some embeddings of strongly positive representations, which are crucial for our classification. These embeddings allow us to realize a strongly positive representation as a (unique irreducible) subrepresentation of a parabolically induced representation of a special type. In this section, we also prove some results concerning the intertwining operators.

In Section 4, we classify irreducible strongly positive representations whose cuspidal support on a two-fold cover of the general linear group-side consists only of the twists of one irreducible self-dual cuspidal representation. This is done by further analysis of the embeddings introduced in the previous section, which enables us to describe them in a more appropriate way. Important properties which are obtained by this analysis allow us to show the uniqueness of such embeddings. In the fifth section, using the same ideas as in the fourth section, we obtain our classification for general irreducible strongly positive representations.

For the convenience of the reader, we cite the main classifications here.

We write ν for the character of $GL(n,F)$ defined by $|\det|_F$, where F is a local non-Archimedean field of a characteristic different than two. We denote by $\widetilde{GL(n,F)}$ a two-fold cover of the general linear group $GL(n,F)$. Let σ denote an irreducible representation of metaplectic group $\widetilde{Sp(n)}$, which is as a set equal to $Sp(n,F)\times\mu_2$, where $\mu_2=\{1,-1\}$. We assume that σ is genuine, i.e., does not factor through μ_2. A representation σ is said to be strongly positive discrete series if for each embedding of the form $\sigma\hookrightarrow\nu^{s_1}\rho_1\times\cdots\times\nu^{s_m}\rho_m\rtimes\sigma_{cusp}$, where $\rho_1,\ldots,\rho_m$ are irreducible genuine cuspidal representations of $\widetilde{GL(n_1,F)},\ldots,\widetilde{GL(n_m,F)}$ and σ_{cusp} is an irreducible genuine cuspidal representation of metaplectic group, we have $s_i>0$ for $i=1,\ldots,m$.

For an irreducible genuine unitary representation ρ of some $\widetilde{GL(n,F)}$ and real numbers a and b such that $b-a$ is a non-negative integer, we call the set $\Delta=\{\nu^a\rho,\nu^{a+1}\rho,\ldots,\nu^b\rho\}$ a genuine segment. We denote by $\delta(\Delta)$ the essentially square-integrable representation attached to the segment Δ (as in [20]). Set $e(\Delta)=\frac{a+b}{2}$. The following theorem describes important embeddings of strongly positive discrete series.

Theorem 1.1. *Let σ be a strongly positive genuine discrete series of some $\widetilde{Sp(m)}$. Then there exists a sequence of genuine segments $\Delta_1,\ldots,\Delta_k$ such that $e(\Delta_1)=\cdots=e(\Delta_{j_1})<e(\Delta_{j_1+1})=\cdots=e(\Delta_{j_2})<\cdots<e(\Delta_{j_s+1})=\cdots=e(\Delta_k)$ and an irreducible genuine cuspidal representation σ_{cusp} of some $\widetilde{Sp(n_{\sigma_{cusp}})}$, such that σ is the unique irreducible subrepresentation of the induced representation $\delta(\Delta_1)\times\cdots\times\delta(\Delta_k)\rtimes\sigma_{cusp}$. (Here we allow $k=0$.)*

Also, if σ can be obtained as an irreducible subrepresentation of some induced representation $\delta(\Delta'_1)\times\cdots\times\delta(\Delta'_l)\rtimes\sigma'_{cusp}$, where $\Delta'_1,\ldots,\Delta'_l$ is a sequence of genuine segments such that $e(\Delta'_1)=\cdots=e(\Delta'_{j'_1})<e(\Delta'_{j'_1+1})=\cdots=e(\Delta'_{j'_2})<\cdots<e(\Delta'_{j'_{s'}+1})=\cdots=e(\Delta'_l)$ and σ'_{cusp} is an irreducible genuine cuspidal representation of some $\widetilde{Sp(n_{\sigma'_{cusp}})}$, then $k=l$, $s=s'$, $j_i=j'_i$ for $i\in\{1,\ldots,s\}$, $\sigma_{cusp}\simeq\sigma'_{cusp}$ and, for $i\in\{1,\ldots,s\}$ and $j_{s+1}=k$, the sequence $\Delta_{j_i+1},\ldots,\Delta_{j_{i+1}-1}$ is a permutation of the sequence $\Delta'_{j_i+1},\ldots,\Delta'_{j_{i+1}-1}$.

Detailed analysis of the embeddings considered in the previous theorem provides additional information about the strongly positive discrete series. The following theorem completes our classification.

Theorem 1.2. *We define a collection of pairs* $(Jord, \sigma')$, *where* σ' *is an irreducible genuine cuspidal representation of some* $\widetilde{Sp(n_{\sigma'})}$ *and Jord has the following form*: $Jord = \bigcup_{i=1}^{k} \bigcup_{j=1}^{k_i} \{(\rho_i, b_j^{(i)})\}$, *where*:

- $\{\rho_1, \rho_2, \dots, \rho_k\}$ *is a (possibly empty) set of mutually non-isomorphic irreducible self-dual cuspidal genuine representations of some* $\widetilde{GL(m_1, F)}, \dots, \widetilde{GL(m_k, F)}$ *such that* $\nu^{a'_{\rho_i}} \rho_i \rtimes \sigma'$ *reduces for* $a'_{\rho_i} > 0$ *(this defines* a'_{ρ_i}*).*
- $k_i = \lceil a'_{\rho_i} \rceil$, *the smallest integer which is not smaller that* a'_{ρ_i}.
- *For each* $i = 1, \dots, k$, $b_1^{(i)}, \dots, b_{k_i}^{(i)}$ *is a sequence of real numbers such that* $a'_{\rho_i} - b_j^{(i)}$ *is an integer, for* $j = 1, 2, \dots, k_i$ *and* $-1 < b_1^{(i)} < b_2^{(i)} < \cdots < b_{k_i}^{(i)}$.

There exists a bijective correspondence between the set of all genuine strongly positive representations and the set of all pairs $(Jord, \sigma')$.

We describe this correspondence more precisely. The pair corresponding to a strongly positive genuine representation σ *will be denoted by* $(Jord(\sigma), \sigma'(\sigma))$.

Suppose that cuspidal support of σ *is contained in the set* $\{\nu^x \rho_1, \dots, \nu^x \rho_k, \sigma_{cusp}\colon x \in \mathbb{R}\}$, *with k minimal (here* ρ_i *denotes an irreducible cuspidal self-dual genuine representation of some* $\widetilde{GL(n_{\rho_i}, F)}$*).*

Let $a'_{\rho_i} > 0$, $i = 1, 2, \dots, k$, *denote the unique positive* $s \in \mathbb{R}$ *such that the representation* $\nu^s \rho_i \rtimes \sigma_{cusp}$ *reduces. Set* $k_i = \lceil a'_{\rho_i} \rceil$. *For each* $i = 1, 2, \dots, k$ *there exists a unique increasing sequence of real numbers* $b_1^{(i)}, b_2^{(i)}, \dots, b_{k_i}^{(i)}$, *where* $a'_{\rho_i} - b_j^{(i)}$ *is an integer, for* $j = 1, 2, \dots, k_i$ *and* $b_1^{(i)} > -1$, *such that* σ *is the unique irreducible subrepresentation of the induced representation*

$$\left(\prod_{i=1}^{k} \prod_{j=1}^{k_i} \delta\left(\left[\nu^{a'_{\rho_i} - k_i + j} \rho_i, \nu^{b_j^{(i)}} \rho_i\right]\right) \right) \rtimes \sigma_{cusp}.$$

Now, $Jord(\sigma) = \bigcup_{i=1}^{k} \bigcup_{j=1}^{k_i} \{(\rho_i, b_j^{(i)})\}$ *and* $\sigma'(\sigma) = \sigma_{cusp}$.

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2. Preliminaries

Let $\widetilde{Sp(n)}$ be the metaplectic group of rank n, the unique non-trivial two-fold central extension of symplectic group $Sp(n, F)$, where F is a non-Archimedean local field of characteristic different from two. In other words, the following holds:

$$1 \to \mu_2 \to \widetilde{Sp(n)} \to Sp(n, F) \to 1,$$

where $\mu_2 = \{1, -1\}$. The multiplication in $\widetilde{Sp(n)}$ (which is as a set given by $Sp(n, F) \times \mu_2$) is given by Rao's cocycle [17]. The topology of the metaplectic groups is explained in detail in [8, Section 3.3].

In this paper we are interested only in genuine representations of $\widetilde{Sp(n)}$ (i.e., those which do not factor through μ_2). So, let $R(n)$ be the Grothendieck group of the category of all admissible genuine representations of finite length of $\widetilde{Sp(n)}$ (i.e., a free abelian group over the set of all irreducible genuine representations of $\widetilde{Sp(n)}$) and define $R = \bigoplus_{n \geqslant 0} R(n)$.

Further, for an ordered partition $s = (n_1, n_2, \dots, n_k)$ of some $m \leqslant n$, we denote by P_s a standard parabolic subgroup of $Sp(n, F)$ (consisting of block upper-triangular matrices), whose Levi factor equals $GL(n_1, F) \times \cdots \times GL(n_k, F) \times Sp(n - |s|, F)$, where $|s| = \sum_{i=1}^{k} n_i$. Then the standard parabolic subgroup $\widetilde{P_s}$ of $\widetilde{Sp(n)}$ is the preimage of P_s in $\widetilde{Sp(n)}$. For the sake of completeness, we explicitly describe

Levi factors of metaplectic groups. Let us denote by $\widetilde{M}_s$ the Levi factor of the parabolic subgroup $\widetilde{P}_s$. There is a natural epimorphism

$$\phi : \widetilde{GL(n_1, F)} \times \cdots \times \widetilde{GL(n_k, F)} \times \widetilde{Sp\big(n - |s|\big)} \to \widetilde{M}_s$$

given by

$$\big([g_1, \epsilon_1], \ldots, [g_k, \epsilon_k], [h, \epsilon]\big) \mapsto \big[(g_1, \ldots, g_k, h), \epsilon_1 \cdots \epsilon_k \epsilon \beta\big],$$

with $\beta = \prod_{i<j}(\det g_i, \det g_j)_F (\prod_{i=1}^{k}(\det g_i, x(h))_F)$, where $x(h)$ is defined in [17, Lemma 5.1], while $(\cdot,\cdot)_F$ denotes the Hilbert symbol of the field F. The Levi factor $\widetilde{M}_s$ differs from the product $\widetilde{GL(n_1, F)} \times \cdots \times \widetilde{GL(n_k, F)} \times \widetilde{Sp(n - |s|)}$ by a finite subgroup, which enables us to write every irreducible genuine representation of $\widetilde{M}_s$ in the form $\pi_1 \otimes \cdots \otimes \pi_k \otimes \sigma$, where the representations $\pi_1, \ldots, \pi_k, \sigma$ are all genuine. The representation of $\widetilde{Sp(n)}$ that is parabolically induced from the representation $\pi_1 \otimes \cdots \otimes \pi_k \otimes \sigma$ will be denoted by $\pi_1 \times \cdots \times \pi_k \rtimes \sigma$.

Let $\widetilde{GL(n, F)}$ be a double cover of $GL(n, F)$, where the multiplication is given by $(g_1, \epsilon_1)(g_2, \epsilon_2) = (g_1 g_2, \epsilon_1 \epsilon_2 (\det g_1, \det g_2)_F)$. Here $\epsilon_i \in \mu_2$, $i = 1, 2$. Here and subsequently, α denotes the character of $\widetilde{GL(n, F)}$ given by $\alpha(g) = (\det g, \det g)_F = (\det g, -1)_F$. For a deeper discussion of the properties of the character α, which is a quadratic character that factors through $GL(n, F)$, we refer the reader to Section 3 of [8] and the references given there.

By ν we mean the character of $GL(k, F)$ defined by $|\det|_F$. Let $\rho_1, \ldots, \rho_n$ denote irreducible cuspidal representations of some $\widetilde{GL(m_1, F)}, \ldots, \widetilde{GL(m_n, F)}$ and σ_{cusp} an irreducible cuspidal representation of some $\widetilde{Sp(k)}$. We say that the representation σ belongs to the set $D(\rho_1, \ldots, \rho_n; \sigma_{cusp})$ if the cuspidal support of σ is contained in the set $\{\nu^x \rho_1, \ldots, \nu^x \rho_n, \sigma_{cusp}\colon x \in \mathbb{R}\}$.

An irreducible representation $\sigma \in R$ is called strongly positive if for each representation $\nu^{s_1}\rho_1 \times \nu^{s_2}\rho_2 \times \cdots \times \nu^{s_k}\rho_k \rtimes \sigma_{cusp}$, where ρ_i, $i = 1, 2, \ldots, k$, are irreducible cuspidal unitary genuine representations, $\sigma_{cusp} \in R$ an irreducible cuspidal representation and $s_i \in \mathbb{R}$, $i = 1, 2, \ldots, k$, such that

$$\sigma \hookrightarrow \nu^{s_1}\rho_1 \times \nu^{s_2}\rho_2 \times \cdots \times \nu^{s_k}\rho_k \rtimes \sigma_{cusp},$$

we have $s_i > 0$ for each i.

Irreducible strongly positive representations are often called strongly positive discrete series.

If ρ is an irreducible genuine unitary cuspidal representation of some $\widetilde{GL(m, F)}$, we say that $\Delta = \{\nu^a\rho, \nu^{a+1}\rho, \ldots, \nu^{a+k}\rho\}$ is a genuine segment, where $a \in \mathbb{R}$ and $k \in \mathbb{Z}_{\geqslant 0}$. Here and subsequently, we abbreviate $\{\nu^a\rho, \nu^{a+1}\rho, \ldots, \nu^{a+k}\rho\}$ as $[\nu^a\rho, \nu^{a+k}\rho]$. If $a > 0$, we call the genuine segment Δ strongly positive. We denote by $\delta(\Delta)$ the unique irreducible subrepresentation of $\nu^{a+k}\rho \times \nu^{a+k-1}\rho \times \cdots \times \nu^a\rho$. $\delta(\Delta)$ is also a genuine, essentially square-integrable representation attached to Δ. Further, let $\widetilde{\Delta} = [\nu^{-a-k}\widetilde{\rho}, \nu^{-a}\widetilde{\rho}]$. Then $\widetilde{\Delta}$ is also a genuine segment and we have $\widetilde{\delta(\Delta)} = \delta(\widetilde{\Delta})$, which follows from [20, Proposition 3.3] and Chapter 4.1 of [8].

For every irreducible genuine cuspidal representation ρ of some $\widetilde{GL(m, F)}$, there exists a unique $e(\rho) \in \mathbb{R}$ such that the representation $\nu^{-e(\rho)}\rho$ is a unitary cuspidal representation. From now on, let $e([\nu^a\rho, \nu^b\rho]) = \frac{a+b}{2}$.

We take a moment to recall a metaplectic version of Tadić's structure formula (Proposition 4.5 from [8]), which enables us to calculate Jacquet modules of an induced representation. Let

$$R^{gen} = \bigoplus_n R\big(\widetilde{GL(n, F)}\big)_{gen},$$

where $R(\widetilde{GL(n, F)})_{gen}$ denotes the Grothendieck group of smooth genuine representations of finite length of $\widetilde{GL(n, F)}$. We denote by m the linear extension to $R^{gen} \otimes R^{gen}$ of parabolic induction from a maximal parabolic subgroup. Let σ denote an irreducible genuine representation of $\widetilde{Sp(n)}$. Then $r_{(k)}(\sigma)$ (the normalized Jacquet module of σ with respect to the standard maximal parabolic subgroup $\widetilde{P}_{(k)}$) can be interpreted as a genuine representation of $\widetilde{GL(k,F)} \times \widetilde{Sp(n-k)}$, i.e., is an element of $R^{gen} \otimes R$. For such σ we can introduce $\mu^*(\sigma) \in R^{gen} \otimes R$ by

$$\mu^*(\sigma) = \sum_{k=0}^{n} \text{s.s.}\big(r_{(k)}(\sigma)\big)$$

(s.s. denotes the semisimplification) and extend μ^* linearly to the whole of R. For $\sigma \in R(n)$ we sometimes write $r_{\widetilde{GL}}(\sigma)$ for $r_{(n)}(\sigma)$.

Using Jacquet modules with respect to the maximal parabolic subgroups of $\widetilde{GL(n,F)}$, we can also define $m^*(\pi) = \sum_{k=0}^{n} \text{s.s.}(r_k(\pi)) \in R^{gen} \otimes R^{gen}$, for an irreducible genuine representation π of $\widetilde{GL(n,F)}$, and then extend m^* linearly to the whole of R^{gen}. Here $r_k(\pi)$ denotes Jacquet module of the representation π with respect to parabolic subgroup whose Levi factor is $\widetilde{GL(k,F)} \times \widetilde{GL(n-k,F)}$. We define $\kappa : R^{gen} \otimes R^{gen} \to R^{gen} \otimes R^{gen}$ by $\kappa(x \otimes y) = y \otimes x$ and extend contragredient $\widetilde{\ }$ to an automorphism of R^{gen} in the natural way. Let $M^* : R^{gen} \to R^{gen}$ be defined by

$$M^* = (m \otimes id) \circ \big(\widetilde{\ }\alpha \otimes m^*\big) \circ \kappa \circ m^*,$$

where $\widetilde{\ }\alpha$ means taking contragredient of the representation and then multiplying by the character α.

The following theorem is fundamental for our calculations with Jacquet modules:

Theorem 2.1. *For $\pi \in R^{gen}$ and $\sigma \in R$, the following structure formula holds*

$$\mu^*(\pi \rtimes \sigma) = M^*(\pi) \rtimes \mu^*(\sigma).$$

Using the previous theorem, we obtain

Lemma 2.2. *Let ρ be a cuspidal genuine representation of $\widetilde{GL(n,F)}$ and $a, b \in \mathbb{R}$ be such that $a + b \in \mathbb{Z}_{\geqslant 0}$. Let σ be an admissible genuine representation of finite length of $\widetilde{Sp(m)}$. Write $\mu^*(\sigma) = \sum_{\pi,\sigma'} \pi \otimes \sigma'$. Then the following hold*:

$$M^*\big(\delta\big(\big[\nu^{-a}\rho, \nu^{b}\rho\big]\big)\big) = \sum_{i=-a-1}^{b} \sum_{j=i}^{b} \delta\big(\big[\nu^{-i}\alpha\widetilde{\rho}, \nu^{a}\alpha\widetilde{\rho}\big]\big) \times \delta\big(\big[\nu^{j+1}\rho, \nu^{b}\rho\big]\big) \otimes \delta\big(\big[\nu^{i+1}\rho, \nu^{j}\rho\big]\big),$$

$$\mu^*\big(\delta\big(\big[\nu^{-a}\rho, \nu^{b}\rho\big]\big) \rtimes \sigma\big) = \sum_{i=-a-1}^{b} \sum_{j=i}^{b} \sum_{\pi,\sigma'} \delta\big(\big[\nu^{-i}\alpha\widetilde{\rho}, \nu^{a}\alpha\widetilde{\rho}\big]\big) \times \delta\big(\big[\nu^{j+1}\rho, \nu^{b}\rho\big]\big) \times \pi \otimes \delta\big(\big[\nu^{i+1}\rho, \nu^{j}\rho\big]\big) \rtimes \sigma'.$$

We omit $\delta([\nu^{x}\rho, \nu^{y}\rho])$ if $x > y$.

The following fact, which follows directly from [8], will be used frequently: for an irreducible genuine representation π of $\widetilde{GL(k,F)}$ and an irreducible genuine representation σ of $\widetilde{Sp(n)}$ in R we have

$$\pi \rtimes \sigma = \alpha\widetilde{\pi} \rtimes \sigma. \tag{1}$$

This important relation can also be obtained through the use of Muić's geometric construction of intertwining operators [16], which is valid in more general cases.

We also use the following equation:

$$m^*\big(\delta\big(\big[\nu^a\rho, \nu^b\rho\big]\big)\big) = \sum_{i=a-1}^{b} \delta\big(\big[\nu^{i+1}\rho, \nu^b\rho\big]\big) \otimes \delta\big(\big[\nu^a\rho, \nu^i\rho\big]\big).$$

Note that the multiplicativity of m^* implies

$$m^*\left(\prod_{j=1}^{n} \delta\big(\big[\nu^{a_j}\rho_j, \nu^{b_j}\rho_j\big]\big)\right) = \prod_{j=1}^{n}\left(\sum_{i_j=a_j-1}^{b_j} \delta\big(\big[\nu^{i_j+1}\rho_j, \nu^{b_j}\rho_j\big]\big) \otimes \delta\big(\big[\nu^{a_j}\rho_j, \nu^{i_j}\rho_j\big]\big)\right). \tag{2}$$

Let us briefly recall the Langlands classification for two-fold covers of general linear groups. As in [9], we favor the subrepresentation version of this classification over the quotient one. This version can be obtained using Lemma 3.1(i) of this paper and part 3 of Proposition 4.2 from [8].

First, for every irreducible essentially square-integrable representation δ of $\widetilde{GL(n, F)}$, there exists an $e(\delta) \in \mathbb{R}$ such that the representation $\nu^{-e(\delta)}\delta$ is unitarizable. Suppose $\delta_1, \delta_2, \ldots, \delta_k$ are irreducible, essentially square-integrable representations of $\widetilde{GL(n_1, F)}, \widetilde{GL(n_2, F)}, \ldots, \widetilde{GL(n_k, F)}$ with $e(\delta_1) \leqslant e(\delta_2) \leqslant \cdots \leqslant e(\delta_k)$. Then the induced representation $\delta_1 \times \delta_2 \times \cdots \times \delta_k$ has a unique irreducible subrepresentation, which we denote by $L(\delta_1, \delta_2, \ldots, \delta_k)$. This irreducible subrepresentation is called the Langlands subrepresentation, and it appears with the multiplicity one in $\delta_1 \times \delta_2 \times \cdots \times \delta_k$. Every irreducible representation π of $\widetilde{GL(n, F)}$ is isomorphic to some $L(\delta_1, \delta_2, \ldots, \delta_k)$. Given π, the representations $\delta_1, \delta_2, \ldots, \delta_k$ are unique up to a permutation. If $i_1, i_2, \ldots, i_k$ is a permutation of $1, 2, \ldots, k$ such that the representations $\delta_{i_1} \times \delta_{i_2} \times \cdots \times \delta_{i_k}$ and $\delta_1 \times \delta_2 \times \cdots \times \delta_k$ are isomorphic, we also write $L(\delta_{i_1}, \delta_{i_2}, \ldots, \delta_{i_k})$ for $L(\delta_1, \delta_2, \ldots, \delta_k)$.

3. Embeddings of strongly positive representations and intertwining operators

In this section we investigate certain embeddings of strongly positive discrete series, which represent the basis of our classification. The main results of this section enable us to study strongly positive discrete series using parabolically induced representations of a special type. We apply ideas and adapt some proofs from Sections 3 and 7 of [6] to our situation and the metaplectic case, and give them here.

We first briefly discuss some intertwining operators. The following lemma is analogous to Theorem 2.6 in [6].

Lemma 3.1. *Assume that $\pi_1, \ldots, \pi_k$ are irreducible genuine representations of $\widetilde{GL(m_1, F)}, \ldots, \widetilde{GL(m_k, F)}$ and σ an irreducible genuine representation of $\widetilde{Sp(n)}$. Let $m = m_1 + \cdots + m_k$ and $l = m + n$. Then the following hold*:

- (i) *Every irreducible quotient of $\pi_1 \times \pi_2 \times \cdots \times \pi_k$ is an irreducible subrepresentation of $\pi_k \times \pi_{k-1} \times \cdots \times \pi_1$. In particular, $\mathrm{Hom}_{\widetilde{GL(m,F)}}(\pi_1 \times \pi_2 \times \cdots \times \pi_k, \pi_k \times \pi_{k-1} \times \cdots \times \pi_1) \neq 0$.*
- (ii) *Every irreducible quotient of $\pi_1 \times \pi_2 \times \cdots \times \pi_k \rtimes \sigma$ is an irreducible subrepresentation of $\alpha\widetilde{\pi}_1 \times \alpha\widetilde{\pi}_2 \times \cdots \times \alpha\widetilde{\pi}_k \rtimes \sigma$. In particular, $\mathrm{Hom}_{\widetilde{Sp(l,F)}}(\pi_1 \times \pi_2 \times \cdots \times \pi_k \rtimes \sigma, \alpha\widetilde{\pi}_1 \times \alpha\widetilde{\pi}_2 \times \cdots \times \alpha\widetilde{\pi}_k \rtimes \sigma) \neq 0$.*

Proof. Claim (i) follows from [8], by repeated application of Propositions 4.1 and 4.3 of that paper. Let us comment on the proof of (ii). Let τ be an irreducible quotient of the representation $\pi_1 \times \pi_2 \times \cdots \times$

$\pi_k \rtimes \sigma$. Then $\widetilde{\tau} \hookrightarrow \widetilde{\pi}_1 \times \widetilde{\pi}_2 \times \cdots \times \widetilde{\pi}_k \rtimes \widetilde{\sigma}$. It is well known that the group $GSp(l)$ acts on $\widetilde{Sp(l)}$ [13, II.1(3)]. As in Section 4 of [5], we choose an element $\eta' = (1, \eta) \in GSp(l)$, where $\eta \in GSp(l')$ is an element with similitude equal to -1. The action of such an element of the group $GSp(l)$ on $\widetilde{Sp(l)}$ extends to the action on irreducible representations, which is (by [13, page 92]) equivalent to taking contragredients. Thus, we obtain the inclusion

$$\widetilde{\tau}^{\eta'} \hookrightarrow \alpha\widetilde{\pi}_1 \times \alpha\widetilde{\pi}_2 \times \cdots \times \alpha\widetilde{\pi}_k \rtimes \widetilde{\sigma}^{\eta}.$$

Since $\widetilde{\sigma}^{\eta} \simeq \sigma$, we have

$$\tau \hookrightarrow \alpha\widetilde{\pi}_1 \times \alpha\widetilde{\pi}_2 \times \cdots \times \alpha\widetilde{\pi}_k \rtimes \sigma.$$

This completes the proof. $\square$

Now we turn our attention to embeddings of strongly positive discrete series. The following lemma [8, Proposition 4.4] ensures the existence of embeddings of irreducible genuine representations:

Lemma 3.2. *For an irreducible representation $\sigma \in R$, there exists an irreducible genuine cuspidal representation $\rho_1 \otimes \rho_2 \otimes \cdots \otimes \rho_k \otimes \sigma_{cusp}$ of some $\widetilde{M_s}$, where $s = (n_1, n_2, \ldots, n_k)$, ρ_i is a genuine irreducible cuspidal representation of $\widetilde{GL(n_i, F)}$, $i = 1, 2, \ldots, k$ and $\sigma_{cusp} \in R$ is an irreducible cuspidal representation such that*

$$\sigma \hookrightarrow \rho_1 \times \rho_2 \times \cdots \times \rho_k \rtimes \sigma_{cusp}.$$

The following theorem provides very useful embeddings of strongly positive discrete series and gives their classifying data.

Theorem 3.3. *Let $\sigma \in R(n)$ denote a strongly positive discrete series. Then there exists a sequence of strongly positive genuine segments $\Delta_1, \Delta_2, \ldots, \Delta_k$ satisfying $0 < e(\Delta_1) \leqslant e(\Delta_2) \leqslant \cdots \leqslant e(\Delta_k)$ (we allow $k = 0$ here) and an irreducible cuspidal representation $\sigma_{cusp} \in R$ such that we have the following embedding*

$$\sigma \hookrightarrow \delta(\Delta_1) \times \delta(\Delta_2) \times \cdots \times \delta(\Delta_k) \rtimes \sigma_{cusp}.$$

Proof. Using the previous lemma, we get the embedding $\sigma \hookrightarrow \rho_1 \times \rho_2 \times \cdots \times \rho_l \rtimes \sigma_{cusp}$; suppose $\sigma_{cusp} \in R(n')$. We consider all possible embeddings of the form

$$\sigma \hookrightarrow \delta(\Delta_1) \times \delta(\Delta_2) \times \cdots \times \delta(\Delta_m) \rtimes \sigma_{cusp},$$

where $\Delta_1 + \Delta_2 + \cdots + \Delta_m = \{\rho_1, \rho_2, \ldots, \rho_l\}$, viewed as an equality of multisets.

Each $\delta(\Delta_i)$ is an irreducible genuine representation of some $\widetilde{GL(n_i, F)}$ (this defines n_i), for $i = 1, 2, \ldots, m$. To every such embedding we attach an $n - n'$-tuple $(e(\Delta_1), \ldots, e(\Delta_1), e(\Delta_2), \ldots, e(\Delta_2), \ldots, e(\Delta_m), \ldots, e(\Delta_m)) \in \mathbb{R}^{n-n'}$, where $e(\Delta_i)$ appears n_i times.

Denote by

$$\sigma \hookrightarrow \delta(\Delta'_1) \times \delta(\Delta'_2) \times \cdots \times \delta(\Delta'_{m'}) \rtimes \sigma_{cusp} \tag{3}$$

a minimal such embedding with respect to the lexicographic ordering on $\mathbb{R}^{n-n'}$ (finiteness of the set of such embeddings gives the existence of a minimal one). Obviously, $e(\Delta'_i) > 0$, for $i = 1, 2, \ldots, m'$. In the following, we show $e(\Delta'_1) \leqslant e(\Delta'_2) \leqslant \cdots \leqslant e(\Delta'_{m'})$. To do this, suppose that $e(\Delta'_j) > e(\Delta'_{j+1})$ for some $1 \leqslant j < m' - 1$.

Lemma 3.1 provides an intertwining operator $\delta(\Delta'_j) \times \delta(\Delta'_{j+1}) \to \delta(\Delta'_{j+1}) \times \delta(\Delta'_j)$, which gives the following maps

$$\begin{aligned}\sigma &\hookrightarrow \delta(\Delta'_1) \times \cdots \times \delta(\Delta'_j) \times \delta(\Delta'_{j+1}) \times \cdots \times \delta(\Delta'_{m'}) \rtimes \sigma_{cusp} \\ &\to \delta(\Delta'_1) \times \cdots \times \delta(\Delta'_{j+1}) \times \delta(\Delta'_j) \times \cdots \times \delta(\Delta'_{m'}) \rtimes \sigma_{cusp}.\end{aligned}$$

The minimality of the embedding (3) implies that σ is in the kernel of previous intertwining operator. The existence of a non-zero kernel, together with Propositions 4.2 and 4.3 from [8], yields that the segments Δ'_j and Δ'_{j+1} are connected in the sense of Zelevinsky. So, we can write $\Delta'_j = [\nu^{a_j}\rho, \nu^{b_j}\rho]$, $\Delta'_{j+1} = [\nu^{a_{j+1}}\rho, \nu^{b_{j+1}}\rho]$, where $0 < a_{j+1} < a_j < b_{j+1} < b_j$, and $\rho \simeq \rho_i$ for some $1 \leqslant i \leqslant l$. Now, using [20], we obtain that the kernel of previous intertwining operator is isomorphic to

$$\delta(\Delta'_1) \times \cdots \times \delta([\nu^{a_j}\rho, \nu^{b_{j+1}}\rho]) \times \delta([\nu^{a_{j+1}}\rho, \nu^{b_j}\rho]) \times \cdots \times \delta(\Delta'_{m'}) \rtimes \sigma_{cusp}. \tag{4}$$

Since $e([\nu^{a_j}\rho, \nu^{b_{j+1}}\rho]) < e(\Delta_j)$, the minimality of the embedding (3) implies that σ is not a subrepresentation of the representation (4). This contradicts our assumption and proves the theorem. □

We proceed by investigating further properties of the obtained embeddings.

Theorem 3.4. *Let $\Delta_1, \Delta_2, \ldots, \Delta_k$ denote a sequence of strongly positive genuine segments satisfying $0 < e(\Delta_1) \leqslant e(\Delta_2) \leqslant \cdots \leqslant e(\Delta_k)$ (we allow $k = 0$ here). Let σ_{cusp} be an irreducible cuspidal genuine representation of $\widetilde{Sp(n)}$. Then the induced representation $\delta(\Delta_1) \times \delta(\Delta_2) \times \cdots \times \delta(\Delta_k) \rtimes \sigma_{cusp}$ has a unique irreducible subrepresentation, which we denote by $\delta(\Delta_1, \ldots, \Delta_k; \sigma_{cusp})$. Also, $\delta(\Delta_1, \ldots, \Delta_k; \sigma_{cusp}) \hookrightarrow \delta(\Delta_1) \rtimes \delta(\Delta_2, \ldots, \Delta_k; \sigma_{cusp})$.*

Proof. We assume that $k > 0$ (otherwise all claims are trivially true) and write $\Delta_i = [\nu^{a_i}\rho_i, \nu^{b_i}\rho_i]$, $i = 1, 2, \ldots, k$. Clearly, the strong positivity of these segments implies $0 < a_i \leqslant b_i$. Further, let us introduce positive integers $j_1 < j_2 < \cdots < j_s$ by

$$\begin{aligned}e(\Delta_1) = \cdots = e(\Delta_{j_1}) &< e(\Delta_{j_1+1}) = \cdots = e(\Delta_{j_2}) \\ &< \cdots < e(\Delta_{j_s+1}) = \cdots = e(\Delta_k).\end{aligned}$$

It follows immediately that the representation

$$\delta(\Delta_1) \times \cdots \times \delta(\Delta_{j_1}) \otimes \delta(\Delta_{j_1+1}) \times \cdots \times \delta(\Delta_{j_2}) \otimes \cdots \otimes \sigma_{cusp} \tag{5}$$

is irreducible, and we show that it appears with multiplicity one in the Jacquet module of $\delta(\Delta_1) \times \delta(\Delta_2) \times \cdots \times \delta(\Delta_k) \rtimes \sigma_{cusp}$ with respect to the appropriate parabolic subgroup. This immediately proves the theorem. We prove this claim using induction over k. We start with the case $k = 1$.

From Lemma 2.2 we get

$$\begin{aligned}\mu^*(\delta(\Delta_1) \rtimes \sigma_{cusp}) = \sum_{i=a_1-1}^{b_1} \sum_{j=i}^{b_1} &\delta([\nu^{-i}\alpha\widetilde{\rho_1}, \nu^{-a_1}\alpha\widetilde{\rho_1}]) \times \delta([\nu^{j+1}\rho_1, \nu^{b_1}\rho_1]) \\ &\otimes \delta([\nu^{i+1}\rho_1, \nu^{j}\rho_1]) \rtimes \sigma_{cusp}.\end{aligned}$$

Therefore, there exist i and j, $a_1 - 1 \leqslant i \leqslant j \leqslant b_1$, such that $\delta(\Delta_1) \otimes \sigma_{cusp} \leqslant \delta([\nu^{-i}\alpha\widetilde{\rho_1}, \nu^{-a_1}\alpha\widetilde{\rho_1}]) \times \delta([\nu^{j+1}\rho_1, \nu^{b_1}\rho_1]) \otimes \delta([\nu^{i+1}\rho_1, \nu^{j}\rho_1]) \rtimes \sigma_{cusp}$ (recall that σ_{cusp} is a cuspidal representation). Of course, we obtain $i = j$, while the strong positivity of the segment Δ_1 implies $-i > -a_1$, i.e., $i = a_1 - 1$. So, $\delta(\Delta_1) \otimes \sigma_{cusp}$ appears with multiplicity one in $\mu^*(\delta(\Delta_1) \rtimes \sigma_{cusp})$.

Now, suppose that claim holds for all numbers less than k. We prove it for k.

Exactness and transitivity of Jacquet modules imply that for every irreducible subquotient of the form (5) of the Jacquet module of the representation $\delta(\Delta_1) \times \delta(\Delta_2) \times \cdots \times \delta(\Delta_k) \rtimes \sigma_{cusp}$ with respect to the appropriate parabolic subgroup, there is some irreducible representation π such that

$$\mu^*\big(\delta(\Delta_1) \times \delta(\Delta_2) \times \cdots \times \delta(\Delta_k) \rtimes \sigma_{cusp}\big) \geqslant \delta(\Delta_1) \times \cdots \times \delta(\Delta_{j_1}) \otimes \pi, \tag{6}$$

where the Jacquet module of π with respect to the appropriate parabolic subgroup contains the representation $\delta(\Delta_{j_1+1}) \times \cdots \times \delta(\Delta_{j_2}) \otimes \cdots \otimes \delta(\Delta_{j_s+1}) \times \cdots \times \delta(\Delta_k) \otimes \sigma_{cusp}$.

Now we take a closer look at the inequality (6). Applying Lemma 2.2, we see that there are $a_i - 1 \leqslant x_i \leqslant y_i \leqslant b_i$, $i = 1, 2, \ldots, k$, such that the following inequality holds:

$$\prod_{i=1}^{k}\big(\delta\big(\big[\nu^{-x_i}\alpha\widetilde{\rho}_i, \nu^{-a_i}\alpha\widetilde{\rho}_i\big]\big) \times \delta\big(\big[\nu^{y_i+1}\rho_i, \nu^{b_i}\rho_i\big]\big)\big) \geqslant \prod_{l=1}^{j_1}\delta\big(\big[\nu^{a_l}\rho_l, \nu^{b_l}\rho_l\big]\big). \tag{7}$$

Because of the irreducibility of the right-hand side, we may assume $a_1 \leqslant a_2 \leqslant \cdots \leqslant a_{j_1}$. Hence, the equality $e(\Delta_1) = e(\Delta_2) = \cdots = e(\Delta_{j_1})$ yields $b_1 \geqslant b_2 \geqslant \cdots \geqslant b_{j_1}$. Comparing the cuspidal supports of both sides of the inequality (7), we obtain the following equality of multisets:

$$\sum_{i=1}^{k}\big(\big[\nu^{-x_i}\alpha\widetilde{\rho}_i, \nu^{-a_i}\alpha\widetilde{\rho}_i\big] + \big[\nu^{y_i+1}\rho_i, \nu^{b_i}\rho_i\big]\big) = \sum_{l=1}^{j_1}\big[\nu^{a_l}\rho_l, \nu^{b_l}\rho_l\big]. \tag{8}$$

The positivity of observed segments forces $a_l > 0$ for every l. We thus get $x_i = a_i - 1$ for every $i = 1, 2, \ldots, k$, so each segment $[\nu^{-x_i}\alpha\widetilde{\rho}_i, \nu^{-a_i}\alpha\widetilde{\rho}_i]$, $i = 1, 2, \ldots, k$, is empty.

Since the representation $\nu^{a_1}\rho_1$ appears on the right-hand side of (8), it must appear on the left-hand side. Since a_1 is the lowest exponent on the right-hand side, we obtain that there is some $1 \leqslant i \leqslant k$ such that $y_i + 1 = a_1$ and $\rho_i \simeq \rho_1$. Observe that this implies $a_i \leqslant a_1$. From this it may be concluded that segment $[\nu^{a_1}\rho_1, \nu^{b_i}\rho_1]$ appears on the left-hand side of (8), so it has to appear on the right-hand side. Since b_1 is the largest exponent there, we get $b_i \leqslant b_1$. We claim that $b_i = b_1$.

On the contrary, suppose that $b_i < b_1$. Then we must have $e(\Delta_i) = \frac{a_i+b_i}{2} < \frac{a_1+b_1}{2} = e(\Delta_1)$, which contradicts the assumption of the theorem.

In this way we get that the first non-empty segment on the left-hand side of (8) equals the first segment on the right-hand side. After canceling this segments on both sides, we continue in the same fashion to obtain $x_i = a_i - 1$ and $y_i = b_i$, for $i > j_1$. Thus, $\delta(\Delta_1) \times \cdots \times \delta(\Delta_{j_1}) \otimes \pi$ appears in $\mu^*(\delta(\Delta_1) \times \delta(\Delta_2) \times \cdots \times \delta(\Delta_k) \rtimes \sigma_{cusp})$ only as an irreducible subquotient of the representation $\delta(\Delta_1) \times \cdots \times \delta(\Delta_{j_1}) \otimes \delta(\Delta_{j_1+1}) \times \cdots \times \delta(\Delta_k) \rtimes \sigma_{cusp}$.

By an argument similar to that in the proof of Lemma 7.4 from [6], we conclude that the multiplicity of $\delta(\Delta_1) \times \cdots \times \delta(\Delta_{j_1}) \otimes \pi$ in $\mu^*(\delta(\Delta_1) \times \delta(\Delta_2) \times \cdots \times \delta(\Delta_k) \rtimes \sigma_{cusp})$ equals the multiplicity of π in $\delta(\Delta_{j_1+1}) \times \delta(\Delta_{j_1+2}) \times \cdots \times \delta(\Delta_k) \rtimes \sigma_{cusp}$.

Combining (6) with (7), we get $\pi \leqslant \prod_{i=j_1+1}^{k} \delta([\nu^{a_i}\rho_i, \nu^{b_i}\rho_i]) \rtimes \sigma_{cusp}$, i.e., π is a subquotient of the representation $\delta(\Delta_{j_1+1}) \times \delta(\Delta_{j_1+2}) \times \cdots \times \delta(\Delta_k) \rtimes \sigma_{cusp}$, which contains the representation $\delta(\Delta_{j_1+1}) \times \cdots \times \delta(\Delta_{j_2}) \otimes \cdots \otimes \delta(\Delta_{j_s+1}) \times \cdots \times \delta(\Delta_k) \otimes \sigma_{cusp}$ in its Jacquet module. By the inductive assumption, such a representation π appears in $\delta(\Delta_{j_1+1}) \times \delta(\Delta_{j_1+2}) \times \cdots \times \delta(\Delta_k) \rtimes \sigma_{cusp}$ with multiplicity one. This proves our claim, which completes the proof of the theorem. □

Theorems 3.3 and 3.4 may be summarized by saying that each genuine strongly positive discrete series is isomorphic to some $\delta(\Delta_1, \ldots, \Delta_k; \sigma_{cusp})$, the unique irreducible subrepresentation of the parabolically induced representation $\delta(\Delta_1) \times \cdots \times \delta(\Delta_k) \rtimes \sigma_{cusp}$, where $e(\Delta_1) \leqslant \cdots \leqslant e(\Delta_k)$. Further examination of these induced representations results in the classification of strongly positive discrete series, which is given in the following two sections.

4. Classification of strongly positive discrete series: $D(\rho;\sigma_{cusp})$ case

In this section, we give a precise classification of a special case of the strongly positive discrete series, those belonging to the set $D(\rho;\sigma_{cusp})$, where ρ is an irreducible genuine cuspidal representation of $\widetilde{GL(n_\rho, F)}$, while σ_{cusp} is an irreducible cuspidal genuine representation of $\widetilde{Sp(n_{\sigma_{cusp}})}$ (this defines n_ρ and $n_{\sigma_{cusp}}$). The partial cuspidal support of every representation belonging to the set $D(\rho;\sigma_{cusp})$ is the representation σ_{cusp}, while the rest of cuspidal support consists of twists of the representation ρ. We also assume that ρ is self-dual, which yields $\alpha\widetilde{\rho}\simeq\rho$. The results of [7] imply that there is a unique $a\geqslant 0$ such that $\nu^a\rho\rtimes\sigma_{cusp}$ reduces. We fix this non-negative real number a through this section. Let k_ρ denote $\lceil a\rceil$, the smallest integer which is not smaller that a. Observe that $k_\rho\geqslant 0$.

We obtain the classification by using the embeddings of strongly positive representations, which have been described in the previous section. We suppose that $\sigma\in D(\rho;\sigma_{cusp})$ is an irreducible strongly positive genuine representation in the whole section.

First, we prove some technical results related to representations of double-covers of general linear groups, which will be needed in the analysis of embeddings of strongly positive representations. Some of these results are closely related to those in Section 1.3 of [9].

Lemma 4.1. *Let Δ_1 and Δ_2 denote strongly positive genuine segments, $\Delta_1=[\nu^{a_1-1}\rho,\nu^{b_1}\rho]$, $\Delta_2=[\nu^{a_1}\rho,\nu^{b_2}\rho]$, where $b_1<b_2$. Then the representation $\nu^{a_1-1}\rho\times L(\delta(\Delta_1),\delta(\Delta_2))$ is irreducible and isomorphic to the representation $L(\nu^{a_1-1}\rho,\delta(\Delta_1),\delta(\Delta_2))$.*

Proof. Let us denote by π the representation $\nu^{a_1-1}\rho\times L(\delta(\Delta_1),\delta(\Delta_2))$. Obviously, $\pi\hookrightarrow\nu^{a_1-1}\rho\times\delta(\Delta_1)\times\delta(\Delta_2)$.

From [9, Lemma 1.3.1] (or [10, Lemma 3.3]), it follows that the only possible irreducible subquotients of π are

$$\pi_1=L\big(\nu^{a_1-1}\rho,\delta\big(\big[\nu^{a_1-1}\rho,\nu^{b_1}\rho\big]\big),\delta\big(\big[\nu^{a_1}\rho,\nu^{b_2}\rho\big]\big)\big),$$
$$\pi_2=L\big(\nu^{a_1-1}\rho,\delta\big(\big[\nu^{a_1-1}\rho,\nu^{b_2}\rho\big]\big),\delta\big(\big[\nu^{a_1}\rho,\nu^{b_1}\rho\big]\big)\big),$$
$$\pi_3=L\big(\delta\big(\big[\nu^{a_1-1}\rho,\nu^{b_1}\rho\big]\big),\delta\big(\big[\nu^{a_1-1}\rho,\nu^{b_2}\rho\big]\big)\big).$$

The Langlands classification shows that π_1 appears with multiplicity one in π. Therefore, it remains to show that π_2, π_3 do not appear. First we address the case $b_1\geqslant a_1$.

Observe that $\pi_2=\delta([\nu^{a_1-1}\rho,\nu^{b_2}\rho])\times L(\nu^{a_1-1}\rho,\delta([\nu^{a_1}\rho,\nu^{b_1}\rho]))$ and $\pi_3=\delta([\nu^{a_1-1}\rho,\nu^{b_1}\rho])\times\delta([\nu^{a_1-1}\rho,\nu^{b_2}\rho])$.

The inclusion $\pi_2\hookrightarrow\delta([\nu^{a_1-1}\rho,\nu^{b_2}\rho])\times\nu^{a_1-1}\rho\times\delta([\nu^{a_1}\rho,\nu^{b_1}\rho])$ enables us to conclude that $m^*(\pi_2)$ contains $\nu^{a_1-1}\rho\otimes\delta([\nu^{a_1-1}\rho,\nu^{b_2}\rho])\times\delta([\nu^{a_1}\rho,\nu^{b_1}\rho])$.

Suppose that π_2 appears in π. Then $m^*(\pi)$ also contains the above representation. In the appropriate Grothendieck group we have

$$\begin{aligned}\delta\big(\big[\nu^{a_1-1}\rho,\nu^{b_1}\rho\big]\big)\times\delta\big(\big[\nu^{a_1}\rho,\nu^{b_2}\rho\big]\big)&=L\big(\delta\big(\big[\nu^{a_1-1}\rho,\nu^{b_1}\rho\big]\big),\delta\big(\big[\nu^{a_1}\rho,\nu^{b_2}\rho\big]\big)\big)\\&\quad+\delta\big(\big[\nu^{a_1-1}\rho,\nu^{b_2}\rho\big]\big)\times\delta\big(\big[\nu^{a_1}\rho,\nu^{b_1}\rho\big]\big).\end{aligned}$$

Analyzing $m^*(\delta([\nu^{a_1-1}\rho,\nu^{b_1}\rho])\times\delta([\nu^{a_1}\rho,\nu^{b_2}\rho]))$ using formula (2), we conclude that the only term of the form $\nu^{a_1-1}\rho\otimes\theta$ in $m^*(\pi)$ is $\nu^{a_1-1}\rho\otimes L(\delta(\Delta_1),\delta(\Delta_2))$. On the other hand, the only term of this form in $m^*(\pi_2)$ is the irreducible representation $\nu^{a_1-1}\rho\otimes\delta([\nu^{a_1-1}\rho,\nu^{b_2}\rho])\times\delta([\nu^{a_1}\rho,\nu^{b_1}\rho])$. Since $b_1\neq b_2$, these representations are not the same, so π_2 cannot appear as a subquotient of π.

Further, observe that $m^*(\pi_3) \geqslant \delta([\nu^{a_1}\rho, \nu^{b_1}\rho]) \times \delta([\nu^{a_1}\rho, \nu^{b_2}\rho]) \otimes \nu^{a_1-1}\rho \times \nu^{a_1-1}\rho$. Suppose that π_3 is a subquotient of π. Then the multiplicativity of m^* implies that $m^*(L(\delta([\nu^{a_1-1}\rho, \nu^{b_1}\rho]), \delta([\nu^{a_1}\rho, \nu^{b_2}\rho])))$ contains $\delta([\nu^{a_1}\rho, \nu^{b_1}\rho]) \times \delta([\nu^{a_1}\rho, \nu^{b_2}\rho]) \otimes \nu^{a_1-1}\rho$.

Analyzing $m^*(\delta([\nu^{a_1-1}\rho, \nu^{b_1}\rho]) \times \delta([\nu^{a_1}\rho, \nu^{b_2}\rho]))$ again, we conclude that the representation $\delta([\nu^{a_1}\rho, \nu^{b_1}\rho]) \times \delta([\nu^{a_1}\rho, \nu^{b_2}\rho]) \otimes \nu^{a_1-1}\rho$ appears there with multiplicity one. Since it obviously appears in $m^*(\delta([\nu^{a_1-1}\rho, \nu^{b_2}\rho]) \times \delta([\nu^{a_1}\rho, \nu^{b_1}\rho]))$, we get a contradiction, so π_3 is not subquotient of π.

This gives $\pi = \pi_1$ and proves the lemma in this case.

If $b_1 = a_1 - 1$, then $\pi_2 = \pi_3 = \nu^{a_1-1}\rho \times \delta([\nu^{a-1}\rho, \nu^{b_2}\rho])$. In the same manner as before we can see that $\pi = \pi_1$, and the lemma follows. □

Lemma 4.2. *Let $\Delta_1, \Delta_2, \ldots, \Delta_k$ denote genuine segments, such that $e(\Delta_1) \leqslant e(\Delta_2) \leqslant \cdots \leqslant e(\Delta_k)$. Then the contragredient of the representation $L(\delta(\Delta_1), \delta(\Delta_2), \ldots, \delta(\Delta_k))$ is isomorphic to $L(\delta(\widetilde{\Delta_k}), \delta(\widetilde{\Delta_{k-1}}), \ldots, \delta(\widetilde{\Delta_1}))$.*

Proof. Taking contragredients of the inclusion

$$L\big(\delta(\Delta_1), \delta(\Delta_2), \ldots, \delta(\Delta_k)\big) \hookrightarrow \delta(\Delta_1) \times \delta(\Delta_2) \times \cdots \times \delta(\Delta_k),$$

we get that the contragredient of the representation $L(\delta(\Delta_1), \delta(\Delta_2), \ldots, \delta(\Delta_k))$ is an irreducible quotient of the representation $\delta(\widetilde{\Delta_1}) \times \delta(\widetilde{\Delta_2}) \times \cdots \times \delta(\widetilde{\Delta_k})$. Applying Lemma 3.1(i), we get that the contragredient of $L(\delta(\Delta_1), \delta(\Delta_2), \ldots, \delta(\Delta_k))$ can be realized as a subrepresentation of the representation $\delta(\widetilde{\Delta_k}) \times \delta(\widetilde{\Delta_{k-1}}) \times \cdots \times \delta(\widetilde{\Delta_1})$. Since the latter representation contains the unique irreducible subrepresentation $L(\delta(\widetilde{\Delta_k}), \delta(\widetilde{\Delta_{k-1}}), \ldots, \delta(\widetilde{\Delta_1}))$, the lemma follows. □

Proposition 4.3. *Let $\Delta_1, \Delta_2, \ldots, \Delta_k$ denote genuine segments, such that $e(\Delta_1) \leqslant e(\Delta_2) \leqslant \cdots \leqslant e(\Delta_k)$. Further, let $\Delta_i = [\nu^{a_1+i-1}\rho, \nu^{b_i}\rho]$, for $i = 1, 2, \ldots, k$, and $b_1 < b_2 < \cdots < b_k$. Then the representation $\nu^{a_1}\rho \times L(\delta(\Delta_1), \delta(\Delta_2), \ldots, \delta(\Delta_k))$ is irreducible.*

Proof. Let us define $\pi = L(\nu^{a_1}\rho, \delta(\Delta_1), \delta(\Delta_2), \ldots, \delta(\Delta_k))$. Since $e(\nu^{a_1}\rho) \leqslant e(\Delta_1)$, we obtain that π is the unique irreducible subrepresentation of $\nu^{a_1}\rho \times L(\delta(\Delta_1), \delta(\Delta_2), \ldots, \delta(\Delta_k))$. Taking contragredients, we get that $\widetilde{\pi}$ is the unique irreducible quotient of $\nu^{-a_1}\widetilde{\rho} \times L(\delta(\widetilde{\Delta_k}), \delta(\widetilde{\Delta_{k-1}}), \ldots, \delta(\widetilde{\Delta_1}))$.

Since $\delta(\widetilde{\Delta_k}) \times \cdots \times \delta(\widetilde{\Delta_3}) \times L(\delta(\widetilde{\Delta_2}), \delta(\widetilde{\Delta_1}))$ is a subrepresentation of $\delta(\widetilde{\Delta_k}) \times \delta(\widetilde{\Delta_{k-1}}) \times \cdots \times \delta(\widetilde{\Delta_1})$, inducing in stages gives the following inclusion:

$$\nu^{-a_1}\widetilde{\rho} \times L\big(\delta(\widetilde{\Delta_k}), \ldots, \delta(\widetilde{\Delta_1})\big) \hookrightarrow \nu^{-a_1}\widetilde{\rho} \times \delta(\widetilde{\Delta_k}) \times \cdots \times L\big(\delta(\widetilde{\Delta_2}), \delta(\widetilde{\Delta_1})\big). \tag{9}$$

Contragredience and the assumptions on the ends of the segments $\Delta_1, \ldots, \Delta_k$, imply $\nu^{-a_1}\widetilde{\rho} \times \delta(\widetilde{\Delta_i}) \simeq \delta(\widetilde{\Delta_i}) \times \nu^{-a_1}\widetilde{\rho}$, for $i \geqslant 3$. Thus, we conclude that the representation on the right-hand side of (9) is isomorphic to $\delta(\widetilde{\Delta_k}) \times \cdots \times \delta(\widetilde{\Delta_3}) \times \nu^{-a_1}\widetilde{\rho} \times L(\delta(\widetilde{\Delta_2}), \delta(\widetilde{\Delta_1}))$.

Since the representation $\nu^{-a_1}\widetilde{\rho} \times L(\delta(\widetilde{\Delta_2}), \delta(\widetilde{\Delta_1}))$ is isomorphic to the contragredient of the representation $\nu^{a_1}\rho \times L(\delta(\Delta_1), \delta(\Delta_2))$, Lemma 4.1 tells us that we can commute representations $\nu^{-a_1}\widetilde{\rho}$ and $L(\delta(\widetilde{\Delta_2}), \delta(\widetilde{\Delta_1}))$. Here, we have applied [2, Corollary 2.1.13], which holds in greater generality and states that an admissible representation is irreducible if and only if its contragredient is. Combining this with (9), we deduce following inclusions:

$$\begin{aligned}
\nu^{-a_1}\widetilde{\rho} \times L\big(\delta(\widetilde{\Delta_k}), \ldots, \delta(\widetilde{\Delta_1})\big) &\hookrightarrow \delta(\widetilde{\Delta_k}) \times \cdots \times L\big(\delta(\widetilde{\Delta_2}), \delta(\widetilde{\Delta_1})\big) \times \nu^{-a_1}\widetilde{\rho} \\
&\hookrightarrow \delta(\widetilde{\Delta_k}) \times \cdots \times \delta(\widetilde{\Delta_2}) \times \delta(\widetilde{\Delta_1}) \times \nu^{-a_1}\widetilde{\rho}.
\end{aligned}$$

On the other hand, according to Lemma 4.2,

$$\widetilde{\pi} = L\big(\delta(\widetilde{\Delta_k}), \delta(\widetilde{\Delta_{k-1}}), \ldots, \delta(\widetilde{\Delta_1}), \nu^{-a_1}\widetilde{\rho}\big),$$

which implies that $\widetilde{\pi}$ is the unique irreducible subrepresentation of $\nu^{-a_1}\widetilde{\rho} \times L(\delta(\widetilde{\Delta_k}), \ldots, \delta(\widetilde{\Delta_1}))$. Now we are in position to conclude that $\widetilde{\pi}$ is both the unique irreducible quotient and the unique irreducible subrepresentation of $\nu^{-a_1}\widetilde{\rho} \times L(\delta(\widetilde{\Delta_k}), \ldots, \delta(\widetilde{\Delta_1}))$. Since it appears with multiplicity one, we deduce that $\nu^{-a_1}\widetilde{\rho} \times L(\delta(\widetilde{\Delta_k}), \ldots, \delta(\widetilde{\Delta_1}))$ is irreducible.

Taking the contragredient finishes the proof. □

Now we are ready to give a precise description of important embeddings of strongly positive genuine discrete series.

Theorem 4.4. *Let $\sigma \in D(\rho, \sigma_{cusp})$ denote an irreducible strongly positive genuine representation. Let $\Delta_1, \Delta_2, \ldots, \Delta_k$ denote the sequence of strongly positive genuine segments, where $0 < e(\Delta_1) \leqslant e(\Delta_2) \leqslant \cdots \leqslant e(\Delta_k)$, such that σ is the unique irreducible subrepresention of the induced representation $\delta(\Delta_1) \times \delta(\Delta_2) \times \cdots \times \delta(\Delta_k) \rtimes \sigma_{cusp}$ (i.e., $\sigma = \delta(\Delta_1, \Delta_2, \ldots, \Delta_k; \sigma)$). Write $\Delta_i = [\nu^{a_i}\rho, \nu^{b_i}\rho]$. Then, $a_i = a - k + i$ and $b_i < b_{i+1}$. Also, $k \leqslant \lceil a \rceil$.*

Proof. Let us consider first the possibility $a = 0$. The inclusion $\sigma \hookrightarrow \delta([\nu^{a_1}\rho, \nu^{b_1}\rho]) \times \cdots \times \delta([\nu^{a_k}\rho, \nu^{b_k}\rho]) \rtimes \sigma_{cusp}$ gives

$$\sigma \hookrightarrow \nu^{b_1}\rho \times \cdots \times \nu^{a_1}\rho \times \cdots \times \nu^{b_k}\rho \times \cdots \times \nu^{a_k}\rho \rtimes \sigma_{cusp}.$$

By the definition of the segment Δ_k, the representation $\nu^{a_k}\rho \rtimes \sigma_{cusp}$ is irreducible (we have supposed $a = 0$), so (1) leads to $\nu^{a_k}\rho \rtimes \sigma_{cusp} \simeq \nu^{-a_k}\rho \rtimes \sigma_{cusp}$. Strong positivity for σ now shows that $k = 0$. We conclude that if $\rho \rtimes \sigma_{cusp}$ reduces, then the only irreducible strongly positive representation in $D(\rho; \sigma_{cusp})$ is σ_{cusp}. In what follows we assume that the representation $\nu^{a}\rho \rtimes \sigma_{cusp}$ reduces for $a > 0$.

The proof is by induction on k. The case $k = 0$ is clear.

Assume $k = 1$. Then

$$\sigma \hookrightarrow \delta\big(\big[\nu^{a_1}\rho, \nu^{b_1}\rho\big]\big) \rtimes \sigma_{cusp} \hookrightarrow \nu^{b_1}\rho \times \nu^{b_1-1}\rho \times \cdots \times \nu^{a_1}\rho \rtimes \sigma_{cusp}.$$

If $a_1 \neq a$, then (1) implies $\nu^{a_1}\rho \rtimes \sigma_{cusp} \simeq \nu^{-a_1}\rho \times \sigma_{cusp}$. In this way, we obtain the embedding

$$\sigma \hookrightarrow \nu^{b_1}\rho \times \nu^{b_1-1}\rho \times \cdots \times \nu^{-a_1}\rho \rtimes \sigma_{cusp},$$

which contradicts the strong positivity of σ. This implies $a_1 = a$.

We also comment on the case $k = 2$. Now we have $\sigma \hookrightarrow \delta([\nu^{a_1}\rho, \nu^{b_1}\rho]) \times \delta([\nu^{a_2}\rho, \nu^{b_2}\rho]) \rtimes \sigma_{cusp}$. As in the previous case, we conclude $a_2 = a$. Since $\delta(\Delta_2; \sigma_{cusp})$ is a subrepresentation of $\delta([\nu^{a}\rho, \nu^{b_2}\rho]) \rtimes \sigma_{cusp}$, induction in stages gives

$$\delta\big(\big[\nu^{a_1}\rho, \nu^{b_1}\rho\big]\big) \rtimes \delta\big(\big[\nu^{a}\rho, \nu^{b_2}\rho\big]; \sigma_{cusp}\big) \hookrightarrow \delta\big(\big[\nu^{a_1}\rho, \nu^{b_1}\rho\big]\big) \times \delta\big(\big[\nu^{a}\rho, \nu^{b_2}\rho\big]\big) \rtimes \sigma_{cusp}.$$

Since σ is the unique irreducible subrepresentation of $\delta([\nu^{a_1}\rho, \nu^{b_1}\rho]) \times \delta([\nu^{a}\rho, \nu^{b_2}\rho]) \rtimes \sigma_{cusp}$, we deduce $\sigma \hookrightarrow \delta([\nu^{a_1}\rho, \nu^{b_1}\rho]) \rtimes \delta([\nu^{a}\rho, \nu^{b_2}\rho]; \sigma_{cusp})$.

This gives us the following embedding:

$$\sigma \hookrightarrow \delta\big(\big[\nu^{a_1+1}\rho, \nu^{b_1}\rho\big]\big) \times \nu^{a_1}\rho \rtimes \delta\big(\big[\nu^{a}\rho, \nu^{b_2}\rho\big]; \sigma_{cusp}\big).$$

The strong positivity of the representation σ and (1) imply that the representation $\nu^{a_1}\rho \rtimes \delta([\nu^{a}\rho, \nu^{b_2}\rho]; \sigma_{cusp})$ reduces. Since $a_1 > 0$, part (ii) of Proposition 13.1 from [18] forces $a_1 \in \{a - 1, b_2 + 1\}$. Namely, the arguments used there rely on the Jacquet module methods which are applicable for the group $\widetilde{Sp(n)}$. Observe that representation $\delta(\Delta_2;\sigma_{cusp})$ coincides with the generalized Steinberg representation that was studied there.

The assumption $a_1 = b_2 + 1$ implies $e(\Delta_1) > e(\Delta_2)$, which contradicts the assumptions of the theorem. So, $a_1 = a - 1$. It remains to show $b_1 < b_2$. If not, the segments $[\nu^{a-1}\rho, \nu^{b_1}\rho]$ and $[\nu^{a}\rho, \nu^{b_2}\rho]$ would not be linked, which gives the embedding $\sigma \hookrightarrow \delta([\nu^{a}\rho, \nu^{b_2}\rho]) \times \delta([\nu^{a-1}\rho, \nu^{b_1}\rho]) \rtimes \sigma_{cusp}$. We obtain that this is impossible in the same way as in the case $k = 1$.

Suppose that the claim holds for all numbers less than k, where $k \geqslant 3$. We prove it for k.

Since $\sigma \hookrightarrow \delta([\nu^{a_1}\rho, \nu^{b_1}\rho]) \rtimes \delta([\nu^{a_2}\rho, \nu^{b_2}\rho], \dots, [\nu^{a_k}\rho, \nu^{b_k}\rho]; \sigma_{cusp})$, strong positivity for σ implies that the representation $\delta([\nu^{a_2}\rho, \nu^{b_2}\rho], \dots, [\nu^{a_k}\rho, \nu^{b_k}\rho]; \sigma_{cusp})$ is also strongly positive. Since $\delta([\nu^{a_2}\rho, \nu^{b_2}\rho], \dots, [\nu^{a_k}\rho, \nu^{b_k}\rho]; \sigma_{cusp})$ is a subrepresentation of $\delta([\nu^{a_2}\rho, \nu^{b_2}\rho]) \times \cdots \times \delta([\nu^{a_k}\rho, \nu^{b_k}\rho]) \rtimes \sigma_{cusp}$ and $e([\nu^{a_2}\rho, \nu^{b_2}\rho]) \leqslant \cdots \leqslant e([\nu^{a_k}\rho, \nu^{b_k}\rho])$, the inductive assumption implies $a_i = a - k + i$, for $i = 2, \dots, k$, and $b_2 < \cdots < b_k$.

We next determine a_1. There are several possibilities:

(i) $0 < a_1 < a - k + 1$: We shall now use repeatedly the fact that $\nu^{m_1}\rho \times \delta([\nu^{m_2}\rho, \nu^{m_3}\rho])$ for $m_1, m_2, m_3 \in \mathbb{R}$ is irreducible if $m_1 < m_2 - 1 < m_3$, to obtain the following embeddings and isomorphisms:

$$\begin{aligned}
\sigma &\hookrightarrow \delta\big(\big[\nu^{a_1}\rho, \nu^{b_1}\rho\big]\big) \times \delta\big(\big[\nu^{a-k+2}\rho, \nu^{b_2}\rho\big]\big) \times \cdots \times \delta\big(\big[\nu^{a}\rho, \nu^{b_k}\rho\big]\big) \rtimes \sigma_{cusp} \\
&\hookrightarrow \delta\big(\big[\nu^{a_1+1}\rho, \nu^{b_1}\rho\big]\big) \times \nu^{a_1}\rho \times \delta\big(\big[\nu^{a-k+2}\rho, \nu^{b_2}\rho\big]\big) \times \cdots \\
&\quad \times \delta\big(\big[\nu^{a}\rho, \nu^{b_k}\rho\big]\big) \rtimes \sigma_{cusp} \\
&\simeq \delta\big(\big[\nu^{a_1+1}\rho, \nu^{b_1}\rho\big]\big) \times \delta\big(\big[\nu^{a-k+2}\rho, \nu^{b_2}\rho\big]\big) \times \nu^{a_1}\rho \times \cdots \\
&\quad \times \delta\big(\big[\nu^{a}\rho, \nu^{b_k}\rho\big]\big) \rtimes \sigma_{cusp} \\
&\ \ \vdots \\
&\simeq \delta\big(\big[\nu^{a_1+1}\rho, \nu^{b_1}\rho\big]\big) \times \delta\big(\big[\nu^{a-k+2}\rho, \nu^{b_2}\rho\big]\big) \times \cdots \times \delta\big(\big[\nu^{a}\rho, \nu^{b_k}\rho\big]\big) \\
&\quad \times \nu^{a_1}\rho \rtimes \sigma_{cusp} \\
&\simeq \delta\big(\big[\nu^{a_1+1}\rho, \nu^{b_1}\rho\big]\big) \times \delta\big(\big[\nu^{a-k+2}\rho, \nu^{b_2}\rho\big]\big) \times \cdots \times \delta\big(\big[\nu^{a}\rho, \nu^{b_k}\rho\big]\big) \\
&\quad \times \nu^{-a_1}\rho \rtimes \sigma_{cusp} \\
&\hookrightarrow \nu^{b_1}\rho \times \cdots \times \nu^{a}\rho \times \nu^{-a_1}\rho \rtimes \sigma_{cusp}
\end{aligned}$$

which contradicts the strong positivity of σ.

(ii) $a_1 = a - k + 2$: Since $L(\delta(\Delta_2), \dots, \delta(\Delta_k))$ is the unique irreducible subrepresentation of $\delta(\Delta_2) \times \cdots \times \delta(\Delta_k)$, inducing in stages gives

$$L\big(\delta(\Delta_2), \dots, \delta(\Delta_k)\big) \rtimes \sigma_{cusp} \hookrightarrow \delta(\Delta_2) \times \cdots \times \delta(\Delta_k) \rtimes \sigma_{cusp}$$

and

$$\delta(\Delta_1) \times L\big(\delta(\Delta_2), \dots, \delta(\Delta_k)\big) \rtimes \sigma_{cusp} \hookrightarrow \delta(\Delta_1) \times \delta(\Delta_2) \times \cdots \times \delta(\Delta_k) \rtimes \sigma_{cusp}.$$

Now, $\sigma \simeq \delta(\Delta_1, \Delta_2, \dots, \Delta_k; \sigma_{cusp})$ yields

$$\sigma \hookrightarrow \delta(\Delta_1) \times L\big(\delta(\Delta_2), \dots, \delta(\Delta_k)\big) \rtimes \sigma_{cusp}.$$

It follows that σ is subrepresentation of $\delta([\nu^{a-k+3}\rho, \nu^{b_1}\rho]) \times \nu^{a-k+2}\rho \times L(\delta([\nu^{a-k+2}\rho, \nu^{b_2}\rho]), \ldots, \delta([\nu^{a}\rho, \nu^{b_k}\rho])) \rtimes \sigma_{cusp}$.

According to Proposition 4.3, this representation is isomorphic to the representation $\delta([\nu^{a-k+3}\rho, \nu^{b_1}\rho]) \times L(\delta([\nu^{a-k+2}\rho, \nu^{b_2}\rho]), \ldots, \delta([\nu^{a}\rho, \nu^{b_k}\rho])) \times \nu^{a-k+2}\rho \rtimes \sigma_{cusp}$, which is further, because $a-k+2 < a$, isomorphic to $\delta([\nu^{a-k+3}\rho, \nu^{b_1}\rho]) \times L(\delta([\nu^{a-k+2}\rho, \nu^{b_2}\rho]), \ldots, \delta([\nu^{a}\rho, \nu^{b_k}\rho])) \times \nu^{-a+k-2}\rho \rtimes \sigma_{cusp}$.

Since $k-a-2 < 0$, we obtain a contradiction with the strong positivity of the representation σ.

(iii) $a-k+2 < a_1$: The assumption $e(\Delta_1) \leqslant e(\Delta_2)$ gives $b_1 < b_2$. Thus, the segments Δ_1 and Δ_2 are not linked and the representations $\delta(\Delta_1) \times \delta(\Delta_2) \times \cdots \times \delta(\Delta_k) \rtimes \sigma_{cusp}$ and $\delta(\Delta_2) \times \delta(\Delta_1) \times \cdots \times \delta(\Delta_k) \rtimes \sigma_{cusp}$ are isomorphic. Since $e(\Delta_1) \leqslant e(\Delta_3)$, in the same way as before we get

$$\sigma \hookrightarrow \delta(\Delta_2) \rtimes \delta(\Delta_1, \Delta_3, \ldots, \Delta_k; \sigma_{cusp}).$$

By the inductive assumption, the representation $\delta(\Delta_1, \Delta_3, \ldots, \Delta_k; \sigma_{cusp})$ is not strongly positive. It follows that σ is not strongly positive, which is impossible.

Finally, we get $a_1 = a-k+1$.

The assumption $b_1 \geqslant b_2$ leads to a contradiction in the same way as in the case $a-k+2 < a_1$ (because now the segment $[\nu^{a-k+1}\rho, \nu^{b_1}\rho]$ contains the segment $[\nu^{a-k+2}\rho, \nu^{b_2}\rho]$). Thus, b_1 must be less than b_2.

Suppose that the remaining claim of the theorem is false, i.e., suppose $k > \lceil a \rceil$. We have two possibilities:

(i) $a_i = a-k+i$, for $i = 1, 2, \ldots, k$. This gives $a_1 \leqslant 0$. Since σ is a subrepresentation of $\delta(\Delta_1) \times \delta(\Delta_2) \times \cdots \times \delta(\Delta_k) \rtimes \sigma_{cusp}$, we have

$$\sigma \hookrightarrow \nu^{b_1}\rho \times \cdots \times \nu^{a_1}\rho \times \nu^{b_2}\rho \times \cdots \times \nu^{a}\rho \rtimes \sigma_{cusp},$$

contradicting the strong positivity of σ.

(ii) There is some $i \in \{1, 2, \ldots, k\}$ such that $a_i \neq a-k+i$. Let x denote the largest such i. Obviously, σ is a subrepresentation of the induced representation $\delta(\Delta_1) \times \cdots \times \delta(\Delta_{x-1}) \rtimes \delta(\Delta_x, \Delta_{x+1}, \ldots, \Delta_k; \sigma_{cusp})$ (we omit $\delta(\Delta_{x-1})$ if x equals 1). From what has already been proved, we conclude that $\delta(\Delta_x, \Delta_{x+1}, \ldots, \Delta_k; \sigma_{cusp})$ is not strongly positive, contradicting strong positivity of σ.

This completes the proof.

Note that we have actually proved $e(\Delta_1) < e(\Delta_2) < \cdots < e(\Delta_k)$. □

Using the above description of the observed embedding, we prove its uniqueness:

Theorem 4.5. *For an irreducible strongly positive genuine representation $\sigma \in D(\rho; \sigma_{cusp})$, there exist a unique sequence of strongly positive genuine segments $\Delta_1, \Delta_2, \ldots, \Delta_k$, with $0 < e(\Delta_1) \leqslant e(\Delta_2) \leqslant \cdots \leqslant e(\Delta_k)$, and a unique irreducible cuspidal representation $\sigma' \in R$ such that $\sigma \simeq \delta(\Delta_1, \Delta_2, \ldots, \Delta_k; \sigma')$.*

Proof. The uniqueness of the partial cuspidal support implies $\sigma' \simeq \sigma_{cusp}$. Further, suppose that there are two sequences of strongly positive genuine segments, $\Delta_1, \Delta_2, \ldots, \Delta_k$ and $\Delta'_1, \Delta'_2, \ldots, \Delta'_l$, where $e(\Delta_1) \leqslant e(\Delta_2) \leqslant \cdots \leqslant e(\Delta_k)$ and $e(\Delta'_1) \leqslant e(\Delta'_2) \leqslant \cdots \leqslant e(\Delta'_l)$, such that

$$\sigma \hookrightarrow \delta(\Delta_1) \times \delta(\Delta_2) \times \cdots \times \delta(\Delta_k) \rtimes \sigma_{cusp} \tag{10}$$

and

$$\sigma \hookrightarrow \delta\left(\Delta_1'\right) \times \delta\left(\Delta_2'\right) \times \cdots \times \delta\left(\Delta_l'\right) \rtimes \sigma_{cusp}, \tag{11}$$

where σ is the unique irreducible subrepresentation of the above induced representations. Using Theorem 4.4, we show that $k = l$ and $\Delta_i = \Delta_i'$, for $i = 1, 2, \ldots, k$. Observe that the previous theorem implies that we can write $\Delta_i = [\nu^{a-k+i}\rho, \nu^{b_i}\rho]$ and $\Delta_j' = [\nu^{a-l+j}\rho, \nu^{b_j'}\rho]$, where $b_i < b_{i+1}$ and $b_j' < b_{j+1}'$.

First we prove that the right-hand sides in (10) and (11) contain an equal number of segments. Suppose on the contrary, $k \neq l$. There is no loss of generality in assuming $k < l$, which gives $a-k+1 > a-l+1$. From (11) we deduce that the Jacquet module of σ with respect to the appropriate parabolic subgroup has to contain the irreducible representation $\delta(\Delta_1') \otimes \delta(\Delta_2') \otimes \cdots \otimes \delta(\Delta_l') \otimes \sigma_{cusp}$. Now, transitivity and exactness of Jacquet modules, applied to (10), imply that there is some irreducible member $\delta(\Delta_1') \otimes \tau$ of $\mu^*(\delta(\Delta_1) \times \delta(\Delta_2) \times \cdots \times \delta(\Delta_k) \rtimes \sigma_{cusp})$ such that the representation $\delta(\Delta_2') \otimes \cdots \otimes \delta(\Delta_l') \otimes \sigma_{cusp}$ is contained in the Jacquet module of τ.

Lemma 2.2 shows that there are $a-k+i-1 \leqslant x_i \leqslant y_i \leqslant b_i$ such that

$$\prod_{i=1}^{k}\left(\delta\left(\left[\nu^{-x_i}\rho, \nu^{-a+k-i}\rho\right]\right) \times \delta\left(\left[\nu^{y_i+1}\rho, \nu^{b_i}\rho\right]\right)\right) \geqslant \delta\left(\left[\nu^{a-l+1}\rho, \nu^{b_1'}\rho\right]\right).$$

Looking at cuspidal supports on both sides of the previous inequality we get a contradiction, because the representation $\nu^{a-l+1}\rho$ appears on the right-hand side, but the index $a-l+1$ is less then each positive index appearing on the left-hand side. This proves $k = l$.

Further, since the Jacquet module of σ contains the representation $\delta(\Delta_1) \otimes \delta(\Delta_2) \otimes \cdots \otimes \delta(\Delta_k) \otimes \sigma_{cusp}$, there is an irreducible member $\delta(\Delta_1) \otimes \tau_1$ of $\mu^*(\delta(\Delta_1) \times \delta(\Delta_2) \times \cdots \times \delta(\Delta_k) \rtimes \sigma_{cusp})$ such that the Jacquet module of τ_1 with respect to the appropriate parabolic subgroup contains $\delta(\Delta_2) \otimes \cdots \otimes \delta(\Delta_k) \otimes \sigma_{cusp}$. Using Theorem 4.4, it can be proved in a similar way as in the proof of Theorem 3.4 that $\tau_1 \leqslant \delta(\Delta_2) \times \cdots \times \delta(\Delta_k) \rtimes \sigma_{cusp}$, the detailed verification being left to the reader.

In the same way, we conclude that in $\mu^*(\delta(\Delta_1') \times \delta(\Delta_2') \times \cdots \times \delta(\Delta_k') \rtimes \sigma_{cusp})$ there appears an irreducible representation $\delta(\Delta_1) \otimes \tau_1'$ such that Jacquet module of τ_1' with respect to the appropriate parabolic subgroup contains $\delta(\Delta_2) \otimes \cdots \otimes \delta(\Delta_k) \otimes \sigma_{cusp}$. Applying Lemma 2.2 to the right-hand side of (11), we get that there are $a-k+i-1 \leqslant x_i' \leqslant y_i' \leqslant b_i'$ such that

$$\prod_{i=1}^{k}\left(\delta\left(\left[\nu^{-x_i'}\rho, \nu^{-a+k-i}\rho\right]\right) \times \delta\left(\left[\nu^{y_i'+1}\rho, \nu^{b_i'}\rho\right]\right)\right) \geqslant \delta\left(\left[\nu^{a-k+1}\rho, \nu^{b_i}\rho\right]\right).$$

Looking at cuspidal supports on both sides of previous inequality, we deduce that $x_i' = a-k+i-1$. Since $y_i' + 1 > a-k+1$ for $i > 1$, it follows that $y_1' = a-k$. This gives $b_1' \geqslant b_1$. Reversing roles, one gets $b_1 \geqslant b_1'$. It follows that $\Delta_1 = \Delta_1'$.

Also, this yields $\tau_1' \leqslant \delta(\Delta_2') \times \cdots \times \delta(\Delta_k') \rtimes \sigma_{cusp}$ and $\upsilon_1 \leqslant \delta(\Delta_2) \times \cdots \times \delta(\Delta_k) \rtimes \sigma_{cusp}$.

Proceeding in the same way, we see that there is an irreducible representation $\delta(\Delta_2) \otimes \tau_2'$ appearing in $\mu^*(\tau_1')$, such that Jacquet module of τ_2' with respect to the appropriate parabolic subgroup contains the representation $\delta(\Delta_3) \otimes \cdots \otimes \delta(\Delta_k) \otimes \sigma_{cusp}$. Since $\mu^*(\tau_1') \leqslant \mu^*(\delta(\Delta_2') \times \cdots \times \delta(\Delta_k') \rtimes \sigma_{cusp})$, applying Lemma 2.2 again we get $b_2' \leqslant b_2$. Going back to subquotients of Jacquet modules of the representation on the right-hand side of (10), we deduce that in $\mu^*(\upsilon_1')$ there appears an irreducible representation $\delta(\Delta_2') \otimes \upsilon_2'$ such that Jacquet module of τ_2' contains $\delta(\Delta_3') \otimes \cdots \otimes \delta(\Delta_k') \otimes \sigma_{cusp}$. A further application of Lemma 2.2 gives $b_2 \leqslant b_2'$. This implies $\Delta_2 = \Delta_2'$.

We continue in the same fashion to obtain $\Delta_i = \Delta_i'$, for $i = 1, 2, \ldots, k$. This completes the proof. $\square$

Theorems 4.4 and 4.5 may be summarized by saying that to each strongly positive genuine discrete series $\sigma \in D(\rho; \sigma_{cusp})$ we have attached an increasing sequence of real numbers $b_1, b_2, \ldots, b_{k_\rho}$, where

$b_1 > -1$ and $b_i - a$ is an integer for every $i \in \{1, 2, \ldots, k_\rho\}$, such that σ is the unique irreducible subrepresentation of the induced representation

$$\delta\big(\big[\nu^{a-k_\rho+1}\rho, \nu^{b_1}\rho\big]\big) \times \delta\big(\big[\nu^{a-k_\rho+2}\rho, \nu^{b_2}\rho\big]\big) \times \cdots \times \delta\big(\big[\nu^{a}\rho, \nu^{b_{k_\rho}}\rho\big]\big) \rtimes \sigma_{cusp}. \tag{12}$$

Observe that some segments in (12) may be empty, i.e., we allow the situation $b_i < a - k_\rho + i$ for some $i \in \{1, 2, \ldots, k_\rho\}$. The above listed properties of the numbers b_i imply that $b_i < a - k_\rho + i$ is equivalent to $b_i = a - k_\rho + i - 1$. In that case, the representation $\delta([\nu^{a-k_\rho+i}\rho, \nu^{b_i}\rho])$ may be excluded from (12). It is used just to write our classification in a more uniform way. Also, $b_i \geqslant a - k_\rho + i$ forces $b_j \geqslant a - k_\rho + j$ for $j \geqslant i$, while $b_i < a - k_\rho + i$ forces $b_j < a - k_\rho + j$ for $j \leqslant i$.

We denote by $SP(\rho; \sigma_{cusp})$ the set of all strongly positive genuine discrete series in $D(\rho; \sigma_{cusp})$. Also, let $Jord_\rho$ stand for the set of all increasing sequences $b_1, b_2, \ldots, b_{k_\rho}$, where $b_i \in \mathbb{R}$, $b_i - a \in \mathbb{Z}$, for $i = 1, 2, \ldots, k_\rho$, and $-1 < b_1 < b_2 < \cdots < b_{k_\rho}$.

The previous discussion and Theorem 4.5 imply that we have obtained a mapping from $SP(\rho; \sigma_{cusp})$ to $Jord_\rho$. The injectivity of this mapping follows from Theorem 4.4.

In what follows, we prove the surjectivity of this mapping in pretty much the same way as in Chapter 7 of [12].

Let $b'_1, b'_2, \ldots, b'_{k_\rho}$ denote an increasing sequence appearing in $Jord_\rho$. Theorem 3.4 implies that the induced representation

$$\delta\big(\big[\nu^{a-k_\rho+1}\rho, \nu^{b'_1}\rho\big]\big) \times \delta\big(\big[\nu^{a-k_\rho+2}\rho, \nu^{b'_2}\rho\big]\big) \times \cdots \times \delta\big(\big[\nu^{a}\rho, \nu^{b'_{k_\rho}}\rho\big]\big) \rtimes \sigma_{cusp} \tag{13}$$

has a unique irreducible subrepresentation, which we denote by $\sigma_{(b'_1,\ldots,b'_{k_\rho})}$. The desired surjectivity is a direct consequence of the following theorem.

Theorem 4.6. *The representation $\sigma_{(b'_1,\ldots,b'_{k_\rho})}$ is strongly positive.*

Proof. We prove this theorem using a two-fold inductive procedure – the first induction is over the number of non-empty segments appearing in the induced representation (13) and the second induction is over the number of elements of the first non-empty segment (the one with the smallest exponent in the twist of ρ).

If there are no non-empty segments in (13), then $\sigma_{(b'_1,\ldots,b'_{k_\rho})} \simeq \sigma_{cusp}$ and the claim follows. Suppose that the claim holds for less then n non-empty segments appearing in (13). We prove it for n non-empty segments.

First we deal with the case $b'_{k_\rho-n+1} = a - n + 1$. The representation $\delta([\nu^{a-n+2}\rho, \nu^{b'_{k_\rho-n+2}}\rho]) \times \cdots \times \delta([\nu^{a}\rho, \nu^{b'_{k_\rho}}\rho]) \rtimes \sigma_{cusp}$ contains a unique irreducible subrepresentation, which we for simplicity denote by σ'. By the inductive assumption, σ' is strongly positive. Clearly, $\sigma_{(b'_1,\ldots,b'_{k_\rho})} \hookrightarrow \nu^{a-n+1}\rho \rtimes \sigma'$. This implies

$$r_{\widetilde{GL}}(\sigma_{(b'_1,\ldots,b'_{k_\rho})}) \leqslant \big(\nu^{a-n+1}\rho + \nu^{-a+n-1}\rho\big) \times r_{\widetilde{GL}}\big(\sigma'\big). \tag{14}$$

We again proceed inductively, by the number of elements in the segment $[\nu^{a-n+2}\rho, \nu^{b'_{k_\rho-n+2}}\rho]$.

If $a - n + 2 = b'_{k_\rho-n+2}$, then $\sigma_{(b'_1,\ldots,b'_{k_\rho})}$ is a subrepresentation of $\nu^{a-n+1}\rho \times \nu^{a-n+2}\rho \times \delta([\nu^{a-n+3}\rho, \nu^{b'_{k_\rho-n+3}}\rho]) \times \cdots \times \delta([\nu^{a}\rho, \nu^{b'_{k_\rho}}\rho]) \rtimes \sigma_{cusp}$. The representation $\delta([\nu^{a-n+3}\rho, \nu^{b'_{k_\rho-n+3}}\rho]) \times \cdots \times \delta([\nu^{a}\rho, \nu^{b'_{k_\rho}}\rho]) \rtimes \sigma_{cusp}$ has a unique irreducible subrepresentation, which is strongly positive by the inductive assumption, and will be denoted by σ''. Part (i) of Lemma 7.2 from [12] implies that $\sigma_{(b'_1,\ldots,b'_{k_\rho})}$ is the unique irreducible subrepresentation of $\nu^{a-n+1}\rho \times \nu^{a-n+2}\rho \rtimes \sigma''$. We emphasize that the proof of mentioned lemma in [12] relies completely on Jacquet module methods and uses no conjectures, so can be applied in our case. This gives $\sigma_{(b'_1,\dots,b'_{k_\rho})} \hookrightarrow L(\nu^{a-n+1}\rho, \nu^{a-n+2}\rho) \rtimes \sigma''$. Thus, we obtain

$$r_{\widetilde{GL}}(\sigma_{(b'_1,\dots,b'_{k_\rho})}) \leqslant \big(L\big(\nu^{a-n+1}\rho, \nu^{a-n+2}\rho\big) + \nu^{a-n+1}\rho \times \nu^{-a+n-2}\rho + L\big(\nu^{-a+n-2}\rho, \nu^{-a+n-1}\rho\big)\big) \times r_{\widetilde{GL}}\big(\sigma''\big).$$

Combining the previous inequality with (14), we get $r_{\widetilde{GL}}(\sigma_{(b'_1,\dots,b'_{k_\rho})}) \leqslant \nu^{a-n+1}\rho \times r_{\widetilde{GL}}(\sigma')$, which implies that $\sigma_{(b'_1,\dots,b'_{k_\rho})}$ is strongly positive.

Suppose $b'_{k_\rho-n+2} > a-n+2$ and that the unique irreducible subrepresentation of $\nu^{a-n+1}\rho \times \delta([\nu^{a-n+2}\rho, \nu^{b'}\rho]) \times \cdots \times \delta([\nu^{a}\rho, \nu^{b'_{k_\rho}}\rho]) \rtimes \sigma_{cusp}$ is strongly positive for $a-n+3 < b' < b'_{k_\rho-n+2}$. We prove this for $b' = b'_{k_\rho-n+2}$.

We have

$$\begin{aligned}\sigma_{(b'_1,\dots,b'_{k_\rho})} &\hookrightarrow \nu^{a-n+1}\rho \times \nu^{b'_{k_\rho-n+2}}\rho \times \delta\big(\big[\nu^{a-n+2}\rho, \nu^{b'_{k_\rho-n+2}-1}\rho\big]\big) \times \cdots \\ &\quad \times \delta\big(\big[\nu^{a}\rho, \nu^{b'_{k_\rho}}\rho\big]\big) \rtimes \sigma_{cusp} \\ &\simeq \nu^{b'_{k_\rho-n+2}}\rho \times \nu^{a-n+1}\rho \times \delta\big(\big[\nu^{a-n+2}\rho, \nu^{b'_{k_\rho-n+2}-1}\rho\big]\big) \times \cdots \\ &\quad \times \delta\big(\big[\nu^{a}\rho, \nu^{b'_{k_\rho}}\rho\big]\big) \rtimes \sigma_{cusp}.\end{aligned}$$

The previous inductive assumption and part (iii) of Lemma 7.2 from [12] imply $\sigma_{(b'_1,\dots,b'_{k_\rho})} \hookrightarrow \nu^{b'_{k_\rho-n+2}}\rho \rtimes \sigma'''$ for some irreducible strongly positive representation σ'''. This gives

$$r_{\widetilde{GL}}(\sigma_{(b'_1,\dots,b'_{k_\rho})}) \leqslant \big(\nu^{b'_{k_\rho-n+2}}\rho + \nu^{-b'_{k_\rho-n+2}}\rho\big) \times r_{\widetilde{GL}}\big(\sigma'''\big). \tag{15}$$

Since $b'_{k_\rho-n+2} > a-n+1$, from (14) and (15) is easy to conclude that $\sigma_{(b'_1,\dots,b'_{k_\rho})}$ is strongly positive.

Up to now, we have proved our claim in the case when the observed segment $[\nu^{a-n+1}\rho, \nu^{b'_{k_\rho-n+1}}\rho]$ contains only one representation. Suppose that the claim holds if the segment $[\nu^{a-n+1}\rho, \nu^{b'_{k_\rho-n+1}}\rho]$ contains less than m representations, i.e., if $a-n+1+m > b'_{k_\rho-n+1}$. We prove it for $b'_{k_\rho-n+1} = a-n+1+m$. In that case, $\sigma_{(b'_1,\dots,b'_{k_\rho})}$ can be written as a subrepresentation of $\delta([\nu^{a-n+m}\rho, \nu^{a-n+1+m}\rho]) \times \delta([\nu^{a-n+1}\rho, \nu^{a-n+m-1}\rho]) \times \delta([\nu^{a-n+2}\rho, \nu^{b'_{k_\rho-n+2}}\rho]) \times \cdots \times \delta([\nu^{a}\rho, \nu^{b'_{k_\rho}}\rho]) \rtimes \sigma_{cusp}$. Part (ii) of Lemma 7.2 from [12] shows that this representation has a unique irreducible subrepresentation. Now, the inductive assumption implies $\sigma_{(b'_1,\dots,b'_{k_\rho})} \hookrightarrow \delta([\nu^{a-n+m}\rho, \nu^{a-n+1+m}\rho]) \rtimes \sigma''''$, where σ'''' is an irreducible strongly positive representation. Looking at Jacquet modules of the representation $\delta([\nu^{a-n+m}\rho, \nu^{a-n+1+m}\rho])$ we may conclude in the same way as before that $\sigma_{(b'_1,\dots,b'_{k_\rho})}$ is strongly positive. This completes the proof. □

5. Classification of strongly positive discrete series: general case

We use the results of the previous section to obtain the classification of general genuine strongly positive discrete series. Proofs of the cases covered in the fourth section help us shorten those in this one.

In this section, $\sigma \in R(n)$ denotes the strongly positive discrete series. Suppose $\sigma \in \widetilde{D(\rho_1, \rho_2}, \ldots, \rho_m; \sigma_{cusp})$, where ρ_i is a self-dual, irreducible, genuine cuspidal representation of $GL(n_i, F)$, for $i = 1, \ldots, m$, $\sigma_{cusp} \in R(n')$ an irreducible genuine cuspidal representation and m minimal. Let $a_{\rho_i} \geqslant 0$ denote the unique non-negative real number such that the representation $\nu^{a_{\rho_i}} \rho_i \rtimes \sigma_{cusp}$ reduces.

The results obtained in the third section show that there exist strongly positive genuine segments $\Delta_1, \Delta_2, \ldots, \Delta_l$ such that $0 < e(\Delta_1) \leqslant e(\Delta_2) \leqslant \cdots \leqslant e(\Delta_l)$ and $\sigma \simeq \delta(\Delta_1, \Delta_2, \ldots, \Delta_l; \sigma_{cusp})$. In the following theorem we describe these segments more precisely.

Theorem 5.1. *Let $\Delta_1, \Delta_2, \ldots, \Delta_l$ be as in the previous discussion. Then the representation $\delta(\Delta_1) \times \cdots \times \delta(\Delta_l) \rtimes \sigma_{cusp}$ is isomorphic to the representation*

$$\left(\prod_{i=1}^{m} \prod_{j=1}^{k_i} \delta\left(\left[\nu^{a_{\rho_i} - k_i + j} \rho_i, \nu^{b_j^{(i)}} \rho_i\right]\right) \right) \rtimes \sigma_{cusp} \tag{16}$$

where $k_i \in \mathbb{Z}_{\geqslant 0}$, $k_i \leqslant \lceil a_{\rho_i} \rceil$, $b_j^{(i)} > 0$ such that $b_j^{(i)} - a_{\rho_i} \in \mathbb{Z}_{\geqslant 0}$, for $i = 1, \ldots, m$, $j = 1, \ldots, k_i$. Also, $b_j^{(i)} < b_{j+1}^{(i)}$ for $1 \leqslant j \leqslant k_i - 1$.

Proof. Let $d \in \{1, \ldots, m\}$ be an arbitrary, but fixed integer. Since the representation $\delta([\nu^{x_1} \rho, \nu^{y_1} \rho]) \times \delta([\nu^{x_2} \rho', \nu^{y_2} \rho'])$ is irreducible if ρ and ρ' are non-isomorphic, the representation $\delta(\Delta_1) \times \cdots \times \delta(\Delta_l) \rtimes \sigma_{cusp}$ is isomorphic to the representation

$$\delta(\Delta_{j_1}) \times \cdots \times \delta(\Delta_{j_{s_1}}) \times \delta(\Delta_{i_1}) \times \cdots \times \delta(\Delta_{i_{s_2}}) \rtimes \sigma_{cusp},$$

where $\{j_1, \ldots, j_{s_1}\} \cup \{i_1, \ldots, i_{s_2}\} = \{1, \ldots, l\}$, $e(\Delta_{i_1}) \leqslant \cdots \leqslant e(\Delta_{i_{s_2}})$, the segments $\Delta_{i_1}, \ldots, \Delta_{i_{s_2}}$ consist of twists of ρ_d, while there are no twists of ρ_d in the segments $\Delta_{j_1}, \ldots, \Delta_{j_{s_1}}$. This yields that σ is the unique irreducible subrepresentation of the representation

$$\delta(\Delta_{j_1}) \times \cdots \times \delta(\Delta_{j_{s_1}}) \rtimes \delta(\Delta_{i_1}, \ldots, \Delta_{i_{s_2}}; \sigma_{cusp}).$$

The strong positivity of σ implies that $\delta(\Delta_{i_1}, \ldots, \Delta_{i_{s_2}}; \sigma_{cusp})$ also has to be strongly positive. Using Theorem 4.4 we get the desired conclusion. $\square$

It is now easy to see that minimality of m implies $a_{\rho_i} > 0$, for $i = 1, 2, \ldots, m$.

Using Theorem 5.1, we can prove the following theorem in the same way as Theorem 4.5, the detailed verification being left to the reader.

Theorem 5.2. *Suppose that the representation σ is isomorphic to both representations $\delta(\Delta_1, \Delta_2, \ldots, \Delta_l; \sigma_{cusp})$ and $\delta(\Delta'_1, \Delta'_2, \ldots, \Delta'_{l'}; \sigma'_{cusp})$, where $\Delta_1, \ldots, \Delta_l$ is a sequence of genuine segments such that $e(\Delta_1) = \cdots = e(\Delta_{j_1}) < e(\Delta_{j_1+1}) = \cdots = e(\Delta_{j_2}) < \cdots < e(\Delta_{j_s+1}) = \cdots = e(\Delta_l)$ and $\sigma_{cusp} \in R$ is an irreducible genuine cuspidal representation. Further, suppose that $\Delta'_1, \ldots, \Delta'_{l'}$ is also a sequence of genuine segments, such that $e(\Delta'_1) = \cdots = e(\Delta'_{j'_1}) < e(\Delta'_{j'_1+1}) = \cdots = e(\Delta'_{j'_2}) < \cdots < e(\Delta'_{j'_{s'}+1}) = \cdots = e(\Delta'_{l'})$ and $\sigma'_{cusp} \in R$ is an irreducible genuine cuspidal representation. Then $l = l'$, $s = s'$, $j_i = j'_i$ for $i \in \{1, \ldots, s\}$, $\sigma_{cusp} \simeq \sigma'_{cusp}$ and, for $i \in \{1, \ldots, s\}$ and $j_{s+1} = l$, the sequence $(\Delta_{j_i+1}, \Delta_{j_i+2}, \ldots, \Delta_{j_{i+1}-1})$ is a permutation of sequence $(\Delta'_{j_i+1}, \Delta'_{j_i+2}, \ldots, \Delta'_{j_{i+1}-1})$.*

Let us denote by SP the set of all strongly positive discrete series in R. Write LJ for the collection of all pairs $(Jord, \sigma')$, where $\sigma' \in R$ is an irreducible cuspidal representation and $Jord$ has the following form: $Jord = \bigcup_{i=1}^{n} \bigcup_{j=1}^{k_i} \{(\rho_i, b_j^{(i)})\}$, where

- $\{\rho_1, \rho_2, \dots, \rho_n\} \subset R^{gen}$ is a (possibly empty) set of mutually non-isomorphic irreducible self-dual cuspidal unitary representations such that $\nu^{a'_{\rho_i}} \rho_i \rtimes \sigma'$ reduces for $a'_{\rho_i} > 0$ (this defines a'_{ρ_i}),
- $k_i = \lceil a'_{\rho_i} \rceil$,
- for each $i = 1, 2, \dots, n$, $b_1^{(i)}, b_2^{(i)}, \dots, b_{k_i}^{(i)}$ is a sequence of real numbers such that $a'_{\rho_i} - b_j^{(i)} \in \mathbb{Z}$, for $j = 1, 2, \dots, k_i$, and $-1 < b_1^{(i)} < b_2^{(i)} < \dots < b_{k_i}^{(i)}$.

Let $(Jord, \sigma')$ denote an element of LJ, where $Jord = \bigcup_{i=1}^{n} \bigcup_{j=1}^{k_i} \{(\rho_i, b_j^{(i)})\}$. Then the induced representation

$$\left(\prod_{i=1}^{n} \prod_{j=1}^{k_i} \delta\big(\big[\nu^{a'_{\rho_i} - k_i + j} \rho_i, \nu^{b_j^{(i)}} \rho_i\big]\big) \right) \rtimes \sigma'$$

has a unique irreducible subrepresentation. In this way, to each element $(Jord, \sigma') \in LJ$ we attach an irreducible genuine representation in R.

According to Theorem 5.1, representation $\sigma \in SP$ may be realized as the unique irreducible subrepresentation of a representation of the form (16). Observe that we may suppose $k_i = \lceil a_{\rho_i} \rceil$ because we are allowed to freely add some empty segments by putting $b_j^{(i)} = a_{\rho_i} - k_i + j - 1$ if necessary. In this way, to a strongly positive discrete series σ we attach a pair $(Jord, \sigma_{cusp}) \in LJ$, where $Jord = \bigcup_{i=1}^{m} \bigcup_{j=1}^{k_i} \{(\rho_i, b_j^{(i)})\}$.

We are ready to state and prove the main result of this paper.

Theorem 5.3. *The maps described above give a bijective correspondence between the sets SP and LJ.*

Proof. Theorems 5.1 and 5.2 imply that we have obtained an injective mapping from SP to LJ. Now we prove its surjectivity.

Let $(Jord, \sigma') \in LJ$, where $Jord = \bigcup_{i=1}^{n} \bigcup_{j=1}^{k_i} \{(\rho_i, b_j^{(i)})\}$. Theorem 3.4 implies that the induced representation

$$\left(\prod_{i=1}^{n} \prod_{j=1}^{k_i} \delta\big(\big[\nu^{a'_{\rho_i} - k_i + j} \rho_i, \nu^{b_j^{(i)}} \rho_i\big]\big) \right) \rtimes \sigma'$$

contains a unique irreducible subrepresentation, which we denote by σ. Suppose that σ is not strongly positive. Then there exists some embedding

$$\sigma \hookrightarrow \nu^{s_1} \rho_{i_1} \times \dots \times \nu^{s_r} \rho_{i_r} \times \dots \times \nu^{s_t} \rho_{i_t} \rtimes \sigma'$$

where $s_r \leqslant 0$. Frobenius reciprocity implies that the representation σ contains $\nu^{s_1} \rho_{i_1} \otimes \dots \otimes \nu^{s_r} \rho_{i_r} \otimes \dots \otimes \nu^{s_t} \rho_{i_t} \otimes \sigma'$ in its Jacquet module.

Clearly, $\rho_{i_r} \in \{\rho_1, \dots, \rho_n\}$. There is no loss of generality in assuming $\rho_{i_r} = \rho_n$. Exactness and transitivity of Jacquet modules, combined with the fact that σ is an irreducible subrepresentation of the induced representation

$$\left(\prod_{i=1}^{n-1} \prod_{j=1}^{k_i} \delta\big(\big[\nu^{a'_{\rho_i} - k_i + j} \rho_i, \nu^{b_j^{(i)}} \rho_i\big]\big) \right) \rtimes \delta\big(\big[\nu^{a'_{\rho_n} - k_n + 1} \rho_n, \nu^{b_1^{(n)}} \rho_n\big], \dots, \big[\nu^{a'_{\rho_n}} \rho_n, \nu^{b_{k_n}^{(n)}} \rho_n\big]; \sigma'\big),$$

imply that $\delta([\nu^{a'_{\rho_n} - k_n + 1} \rho_n, \nu^{b_1^{(n)}} \rho_n], \dots, [\nu^{a'_{\rho_n}} \rho_n, \nu^{b_{k_n}^{(n)}} \rho_n]; \sigma')$ contains a representation of the form $\nu^{s'_1} \rho_n \otimes \dots \otimes \nu^{s_r} \rho_n \otimes \dots \otimes \nu^{s'_{t'}} \rho_n \otimes \sigma'$ in its Jacquet module. Now, using Lemma 26 from [1], which can be applied in our situation (this is explained in full detail in the proof of Lemma 3.1 in [7]), and Frobenius reciprocity, we deduce that $\delta([\nu^{a'_{\rho_n}-k_n+1}\rho_n, \nu^{b_1^{(n)}}\rho_n], \ldots, [\nu^{a'_{\rho_n}}\rho_n, \nu^{b_{k_n}^{(n)}}\rho_n]; \sigma')$ is a subrepresentation of $\nu^{s'_1}\rho_n \times \cdots \times \nu^{s_r}\rho_n \times \cdots \times \nu^{s'_{t'}}\rho_n \rtimes \sigma'$. This contradicts Theorem 4.6 and shows that each element of LJ is attached to some strongly positive discrete series.

The maps described above are obviously inverse to each other. □

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